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A CONSOLIDATED LIST OF HINDU MATHEMATICAL WORKS

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An attempt is made here to bring together the vast majority of known Hindu mathematical treatises. A few of them have seen the light of day, thanks to the marvellously zealous endeavours of scholars; but the bulk of them are known to the expert world only and still remain in the form of dust-laden and worm-eaten MSS., continuing their age-long slumbers in uncared-for corners of archives; while not an inconsiderable number have actually perished, only their memory being kept alive through cross-references in some of the extant works.

All Hindu scientific works can be classified under two heads, *viz.*, Pauruṣéya (personal) and Apauruṣéya (impersonal). The first category contains the treatises composed by human beings; to the works belonging to the second no human authorship can be ascribed; they are supposed to be divine or semi-divine in origin, having been handed down by word of mouth by sages and deities. Of the latter class* only the Sūrya-siddhānta (ascribed to the Sun-god) is preserved to-day; and while all other works of this class have been omitted in the following list, this one finds a place since it has inspired so many other major treatises and commentaries and is even to-day held in high esteem as an authoritative treatise in Upper India. It may be mentioned that according to most scholars, the Sūrya-siddhānta as it exists to-day is not the original one, but only a recent version, revised some time about 900 A.D., of the original.

The dates assigned to the authors and works are in a few cases absolutely precise, the authors themselves having explicitly mentioned the date of composition and even a few auto-biographical details in the body of the

(*Continued on page 70*)

* Thus the Sulva sūtras have not been mentioned. Another omission is the Bhakśāli MS.

Sl. No.	Author's name, date, etc	Name of the work	General nature of the work
1	ĀRYABHATA I (born at Asmaka in 475 A.D.)	Āryabhatīya (composed in 498 A.D.)	Consists of 118 stanzas divided into four parts dealing respectively with : (i) Astronomical constants and table of sines (ii) Principles of elementary algebra and mensuration (iii) Planetary motions (iv) Problems of spherical astronomy
2	BHĀSKARA I (c. 522 A.D.; a pupil of Āryabhata)	(i) Laghu-bhāskarīya (ii) Mahā-bhāskarīya	(i) A work of 215 stanzas in eight chapters dealing with spherical astronomy (ii) A work of about 400 quartrains on the customary topics of spherical astronomy, more exhaustive than the foregoing, all the same a supplement to Āryabhatīya
3	VARĀHAMIHIRA (c. 525 A.D. ; son of Ādityadāsa and a resident of Avanti)	(i) Bṛhat-samhita (ii) Pança-siddhāntika	A general reference-book containing several interesting items of information, including Astronomy. A resumé of five ancient texts due to Sūrya, Paulisa, Romaka, Vasistha and Brahma
4	BRAHMAGUPTA (born 598 A D. ; son of Jishnu)	(i) Brahmasphuta siddhānta (composed in 628 A D) (ii) Khanda-khādyaka (665 A.D.)	(i) A comprehensive treatise on astronomy with a section devoted to Algebra. Some of Āryabhata's ideas are vigorously attacked in this work. (ii) A work in two parts. This betrays no trace of animosity against Āryabhata; even an attempt at reconciliation is made
5	LALLA (c. 748 A.D., son of Bhatta-trivikrama and grandson of Sāmba)	(i) Sisya-dhi-vrddhida (ii) Pātīganitam	(i) A work of the Āryabhata School, the model for Bhāskara II
6	MAHĀVĪRA (c. 853 A.D.)	Ganita-sāra-sangraha	A treatise on Arithmetic, Algebra, etc, with special treatment of indeterminate equations
7	SRĪDHARA (c. 853 A D.)	Trisatikā	An authoritative treatise in 300 āryās

A Consolidated List of Hindu Mathematical Works

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Commentator's name and date	Name, date and general nature of the commentary	Remarks
(i) BHĀSKARA I (522 A.D.)		
(ii) SŪRYADEVA (presumably about 1400 A.D.)	(ii) Āryabhata-sūtrārtha-prakāsika—a work referred to with veneration by Nīlakanta (*infra*)	
(iii) PARAMĒSVARA (1431 A.D.)	(iii) Bhata dīpika	(iii) has been published in Leiden
(iv) NĪLAKANTA (*c.* 1500 A.D.)	(iv) Nīlakantīya—a masterly exposition of the text interspersed liberally with the results of the commentator's own observations. Some of the processes dealt with reveal a strikingly versatile originality and a comprehensive grasp of the principles—and genuine mathematical talent	Nīlakanta was the pupil of one Dāmodara, son of Paramêsvara—all three Nambūdiri Brahmins from Malabar
(v) YALLAYYA (son of Srīdhara)	(v) Tātparya-Vyākhyāna	
(i) PARAMESVARA (1431 A.D.)	(i) Sundarī	(i) Recently edited by Dr. A. N. Singh
(ii) PRABHĀKARA	(ii) Tantra-pradīpa	
(iii) SANKARA (son of Nārāyaṇa)	(iii) Bhāskarīya vivaranam	
(i) PARAMÉSVARA (1431 A.D.)	(i) (*a*) Karma-dīpika—a brief but lucid exposition of the text. (*b*) Siddhānta-dīpika—a more ambitious work than the former	Both the texts along with Paramésvarā's commentary have been published last year by Ānandāshram, Poona
(ii) BHĀSKARA		
(iii) GOVINDASWĀMI	(iii) Prayōga-raçana	
(iv) MAKKI-BHATTA (1377 A.D.)	(iv) Gaṇita vilāsa	
(v) SRIKANTA (pupil of Rudra)		
		Varāha has written also an authoritative work 'Bṛhat-jātaka' on astrology, commented on by Govinda-Bhatt (*c.* 1400 A.D.). His other astrological works are 'Laghujātaka', 'Yoga-yātra and 'Vivāhapatala'.
(i) ÇATURVĒDA-PRITHŪDAKA SWĀMIN (854 A.D.)		Brahmagupta's text was Alberuni's *source for* his study of Hindu mathematics. He translated the work into Persian.
(i) LALLA (748 A.D.)		
(ii) BHATŌTPALA (956 A.D.)		
(iii) ÇHATURVĒDA-PRITHŪDAKA-SWĀMIN (854 A.D.)		
(iv) SOMĒSVARA		
(v) ĀMARĀJA (son of Mahadeva)		
(vi) VARUNA (1040 A.D.)		
(i) MALLIKĀRJUNA SŪRI (1179 A.D.)		The second work is unknown, but it is referred to by Bhāskara II

Sl. No.	Author's name, date, etc.	Name of the work	General nature of the work
8	VITTÉSVARA (*c.* 931 A.D.)	Karanasāra	
9	MUNJĀLA (*c* 932 A.D)	(i) Laghumānasa	A synopsis of astronomy in 60 stanzas
		(ii) Bṛhanmānasa	
10	VIJAYANANDIN (*c.* 966 A.D)	Karanatilaka	
11	ĀRYABHATA II (*c.* 950 A.D.)	Mahāsiddhānta	
12	ANONYMOUS	Sūrya-siddhānta (*c.* 1000 A.D.)	A fairly exhaustive treatise on Astronomy and a good deal of subservient mathematics—a fresh version of the original text
13	SRĪPATHI (*c.* 1000 A.D.)	(i) Siddhānta-ṣékhara (1027 A.D.) (ii) Ganita-tilaka (iii) Dhruvamānasa (1056 A.D.) (iv) Dhīkotida-karanam (1039 A.D.)	(i) A masterly treatise on Astronomy in 20 chapters containing about 900 beautiful verses (iii) 105 stanzas summarising Astronomy, based on Munjāla's work—mentions precession omitted in Siddhāntaṣékhara (iv) 20 stanzas on eclipses
14	BHĀNUBHATTA (*c.* 1040 A.D.)	(i) Karana-paratilaka (ii) Ādityapratāpa-siddhānta (1046 A.D.)	
15	BHŌJARĀJA (Ruler of Dhārānagari, *c.* 1042 A.D.)	Rājamṛgānka	
16	DASABALA (*c.* 1058 A D.)	Karaṇa-kamala-mārtānda	
17	BRAHMADĒVA (*c.* 1112 A.D son of Chandradéva of Muttra)	Karanaprakāsa	A work of the Āryabhata School
18	ṢATĀNANDA (*c* 1099 A.D) (son of Sankara)	(i) Bhāsvati-karaṇa (ii) Karanōthama (1116 A.D.) (iii) Abhilaṣitārtha-cintāmani (1129 A.D.)	
19	MĀDHAVA (*c.* 1100 A.D.)—a native of Sangamagrāma)	Vēnvāroha	Problems on the sun and moon

Commentator's name and date	Name, date and general nature of the commentary	Remarks
(i) SURYADÉVA (son of Somasuta)		Munjāla also deals with precession of the equinoxes
(ii) SRĪKANTA		
(iii) CATURVÉDA-PRTHŪDAKA (864 A.D.)		
(iv) PRASASTIDHARA (958 A.D.)		
(v) PARAMÉSVARA (1431 A.D.)	(v) Pāramēsvari-Vyākhyā	
(vi) YALLAYYA (son of Srīdhara)	(vi) Kalpalatā	
		This has never been very much in vogue
(i) BHĀSKARA I (522 A.D.)	(i) *Obviously a commentary* on the original text	
(ii) THIMMAYAJVĀ	(ii) Kāma-dōgdhi	
(iii) YALLAYYA	(iii) Kalpavati	
(iv) SRĪKANTA (pupil of Rudra)	(iv) Vivarana	
(v) RANGANĀTHA (son of Vallāla)	(v) Gūdhārtha-prakāsini (1603 A.D.)	
(vi) NṚSIMHA (son of Kṛṣna)	(vi) Saurabhadhyam (1611 A.D.)	
(vii) VISVANĀTHA (1628 A.D.)	(vii) Gahanārthaprakāsikā—this work teems with illustrative examples	
(viii) DADHABHĀI DĀNDÉKARA (son of Mādhava)	(viii) Kiranāvali (1719 A.D.)	
(ix) BHŪDHARA (son of Dévadatta)	(ix) Vivarana (1649 A.D.)	
(x) BHAIRAVA	(x) Sūryasiddhānta-jyāganitam	
(i) MAKKIBHATTA (1377 A.D.)	(i) Ganita-bhūsana—incomplete, running only up to the end of Chapter IV—a lucid and well-reasoned-out work.	Srīpathi has the following astrological works also to his credit: (i) Srīpathi-paddhati (ii) Srīpathi-ratnamāla (iii) Ratnasāra (iv) Srīpathi nibandha (v) Srīpathi samuççaya
		MS. available at Jesselmere
		MS. of the text available with me

Sl. No.	Author's name, date, etc.	Name of the work	General nature of the work
20	BHĀSKARA II (*c.* 1150 A.D.; son and disciple of Mahéşvara)	(i) Bījōpanaya (1156 A.D.)	
		(ii) Karaṇa-kutūhala (1183 A.D.)	
		(iii) Siddhānta-şirōmaṇi	(iii) This is by far the most well-known and authoritative work on Hindu mathematics. Every branch, then known, of the subject is dealt with in profuse detail. There are four parts : (*a*) *Lilāvati* :—a section devoted to elementary Algebra and Geometry. It is in the form of a lecture addressed to a little girl, Lilāvati. Every topic in Algebra is touched upon and illustrative examples abound (*b*) Bija-ganita—This is the part dealing with higher Algebra

Commentator's name and date	Name, date and general nature of the commentary	Remarks
(i) CHARKADHARA (son of Vāmana)	(i) Yantra-çintāmani	
(ii) KṚIPĀRĀMA (1498 A.D.)		
(iii) RĀMA (1628 A D.)		
(iv) DINAKARA (1648 A.D.		
(i) VISVANĀTHA (1441 A.D)		
(ii) SŌDHALA (1463 A.D)		
(iii) SUMATIHARṢA (1619 A.D.)		
(iv) PADMANĀBHA (1739 A.D.)		
(v) SUDHĀKARA DVIVÉDI		
(a) (i) RAMAKRṢNA (son of Laksmana)	(i) Lilavathi-vrthi (1678 A.D.)	
(ii) LAKSHMĪDĀSA		
(iii) GŌVARDHANA		
(iv) SRĪKANTA (son of Rudra)		
(v) MOṢADEVA (son of Bhimadeva)		
(vi) PARAMÉSVARA (1431 A.D.)	(vi) An illuminating treatise, like any of his other works	
(vii) RANGANĀTHA	(vii) Vyākyā-vilāsa	
(viii) SRĪDHARA	(viii) Sarva-bōdhini-vyākhyā	
(ix) NĀRĀYANA	(ix) Karmapradīpika	
(x) GANÈṢA (son of Kéṣava)	(x) Buddhi-vilāṣini (1545 A.D)	
(xi) GANÉṢA (son of Dundhirāja)	(xi) Ganita-manjari (1568 A.D.)	
(xii) GHANÉṢVARA	(xii) Līālvati-bhūshanam (1541 A.D.)	
(xiii) MUNIṢVARA (son of Ranganātha)	(xiii) Nisṛstartha-duti or Lilāvati-vivṛti (1635 A.D.)	
(xiv) SŪRYADĀSA (son of Gnānarāja)	(xiv) Ganitāmṛta-kūpikā (1541 A.D.)	
(xv) GANGĀDHARA (son of Gōvardhana)	(xv) Ganitāmṛta-sāgari (1420 A.D.)	
(xvi) RĀMAKRṢNA (son of Sadāsiva)	(xvi) Manōranjanam	
(xvii) RĀMAKRṢNA (son of Ghanasyāma)	(xvii) Manōranjana-tīka (1760 A.D.)	
(xviii) MAHĪDĀSA	(xviii) Lilāvati-tīka (1587 A D.)	
(xix) MAHĪDHARA	(xix) Lilāvati vivarana (1635 A.D.)	
(xx) RANGANĀTHA (son of Nrsimha)	(xx) Mita-bhāṣini (1630 A.D.)	
(xxi) GIRIDHARA		Commentaries (xxi) to (xxv) are in languages other than Sanskrit; the first of these is a Prākrit work
(xxii) SŌMASĀDHU		
(xxiii) UDAYARĀMA		
(xxiv) LĀBHAVARDHANA PRĒMJI		
(xxv) MOKSHADÉVA		
(xxvi) ANONYMOUS	(xxvi) Kriyākramakari (1500 A.D.)	The author seems to have written a commentary on Tantrasangraha also (*infra*)
(xxvii) NĀRĀYANA	(xxvii) Karmapradīpika	
(b) (i) SIDDHĀNTA DAIVAGNA *alias* RĀMAKRṢNA (son of Lakshmana, son of Nṛsimha)	(i) Bījaprabōdha (1687 A.D.)	
(ii) SŪRYADĀSA (son of Gnānarāja)	(ii) Bījabhāshya (1541 A.D.)	

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Sl. No.	Author's name, date, etc.	Name of the work	General nature of the work
			(c) Gaṇitādhyāya—Astronomy proper, celestial mechanics, the usual processes, results and calculations
			(d) Gōlādhyāya—Spherical Astronomy—exposition of the principles established in the last part.
21	BALLĀLASÉNA (c. 1160 A.D ; son of King Lakshmanaséna)	Adbhutasāgaram	Deals with the Samhita portion. It was composed 8 years after the author's succession to the throne. There are two quotations from the present Sūrya-siddhânta ; and Garga and Vṛddhagarga are separately mentioned
22	ANANTADÉVA (c. 1192 .D.)		
23	SRĪDHARĀÇĀRYA (c. 1227 A.D)	Laghukhéçara-siddhi	
24	GOVINDABHATTA *alias* TALAKKULATHUR BHATTATIRI	Dasādhyāyı	A marvellous commentary (as good as a text), on Varāhā's Hōrā-sāstra It runs up to the tenth chapter only.
25	MAHĀDÉVA I (c. 1278 A.D)	(i) Grahasıddhi	
		(ii) Sārani	A work is 42 stānzas, a digest of the classics of Āryabhata, Varāha and others.
		(iii) Mahādévı (1360 A.D.)	
26	NĀRĀYAṆA (son of Nrsimha)	Ganitakaumadi	
27	MAHÉNDRASŪRI (c. 1320 A D , Court-pandıt of Firoz Shah Tuglāq)	Yantrarāja (1370 A.D.)	A work in 182 verses which appears to be a translation of Persian text.
28	ANANTA (c 1357 A.D.)		
29	MAHĀDÉVA II (c. 1367 A D.)	Kāmadhénu karanam	This deals with 'Pançānga's
30		Nārmadam (1378 A.D.)	
31	PADMANĀBHA (c. 1398 A.D)		
32	PARAMÉSVARA (c. 1430 A.D.) He belonged to Ālāthur in Kérala and advanced a system of astronomy embodying corrections for cumulative	(i) Drg-ganita-tantra (1431 A.D.)	The text introduces the new system, put forward after 55 years of active observation.
		(ii) Gōladīpika	A concise review of Spherical Astronomy
		(iii) Karmadīpika	A lucid gloss on Mahabhāskarīya
		(iv) Līlāvati-vyākyā	A commentary on Līlāvati

Commentator's name and date	Name, date and general nature of the commentary	Remarks
(iii) KRṢNA DAIVAGṆA (1602 A·D)	(iii) Bījanavānkura *or* Bīja-pallava *or* Kalpalatāvatāra	
(iv) JANĀRDANA	(iv) Bījōdāharanam	
(v) HARIDĀSA	(v) Vāsanābhāshya	
(vi) KRIPĀRĀMA MISRA (1792 A D.)	(vi) Bījōdāharana-bālabōdhini	
(c) (i) GOPINĀTHA (son of Bhairava)	(i) Siddhānta-suryodaya (1528 A.D.)	
(ii) MUNĪSVARA (son of Ranganātha)	(ii) Marīchi-tīkā (1635 A.D.)	
(iii) LAKSHMIDĀSA	(iii) Ganitatatva-çintāmani (1500 A.D.)	
(iv) NṚSIMHA (son of Krsṇa)	(iv) Vāsanā bhāṣya (1621 A.D.)	
(v) RANGANĀTHA (son of Nrsimha)	(v) Mita-bhāshini (1657 A.D.)	
(d)(i) MUNĪSWARA (1627 A.D.)		
(ii) LAKSHMĪDĀSA (1500 A.D.)		
(iii) NRSIMHA	(iii) Vāsanā-vārtika (1586 A.D.)	
		An incomplete MS. is available in the Government Library, Benares
		The author is almost a legendary figure in Kérala
NRSIMHA (1528 A.D.)		Published by Ānandāshram, Poona.
(i) MALAYĒNDRASŪRI (1378 A.D.), a direct pupil of the author	(i) The commentory contains 1000 'Anushthub' verses. The obliquity is given as 23° 35′.	Edited by Sudhākara Dvivédi in 1882
(ii) YAJNĒSVARA (1842 A.D)		
ANANTADAIVAGNA		
		Published in the Trivandrum Skt. series

No. Sl.	Author's name, date, etc.	Nnme of the work	General nature of the work
	errors; it is in vogue even at the present day	(v) Bhātadīpika	A commentary on Āryabhatīya
		(vi) Laghu-bhāskarīya-tīkā	
		(vii) Laghumānasa-tīkā	An exposition of Munjālā's Laghumānasa
33		Karaṇapaddhati (1431 A.D.?)	A highly advanced work on astronomy and mathematics (literally 'operational technique') in 10 chapters
34	ACHYUTA PISHĀROTI	Kriyākrama	
35	SŪRYADÉVA YAJVĀ	(i) Āryabhata-sūtrārtha-prakāsikā	A succinct commentary on Āryabhatīya
		(ii) Laghumānasa-vyākhyā	
36	DĀMŌDARA (c. 1457 A.D.; son of Paraméṣvara)	(i) Muhūrthābharana (ii) Bhadradīpaka	A work on general astronomy in Malayālam verse
37	GANGĀDHARA (c. 1434 A.D.)	Chandramāna	A work based on the Sūrya-siddhānta
38	MAKARANDA (c. 1438 A.D.)	Makaranda	Also belonging to the Soura School; very much in vogue in Oudh and nearby places
39	KÉSAVA (c. 1456 A.D.; son of Kamaḷākara, pupil Vaidyanātha and father of Ganésa)	(i) Grahakauthukam (ii) Varshagrahasiddhi (iii) Tithisiddhi (iv) Ganitadīpika (v) Siddhānta-vāsanāpatha	
40	VIHAṆA (c. 1456 A.D.)	(i) Vārshikatantra	
41		(ii) Vākyakaraṇa	
42	NĪLAKANTA (c. 1500 A.D.; pupil of Damodara)	(i) Āryabhatīya-bhāṣya	The inimitable commentary on Āryabhatīya
		(ii) Tantra-sangraha	Another masterpiece of the author: 'Science Digest'
		(iii) Gōlasāra (iv) Siddhānta-darpaṇa	
43	LAKSHMIDĀSA (c. 1500 A.D.)	Ganitatantra-çintāmani	An exposition on Siddhānta Siromani
44	GNĀNARĀJA (c. 1503 A.D.)	(i) Siddhānta-sundaram (ii) Bījagaṇita	A work in 2 parts, Gōlādhyāya and Ganitādhyāya
45	GAṆÉSA (c. 1507 A.D. son and pupil of Késava)	(i) Grahalāghava	Composed when the author was 13
		(ii) Laghutithi-çintāmani (iii) Brhat-tithi-çintāmani (iv) Līlavati-tīka (v) Siddhānta-sirōmani tīka	
46	ANANTADAIVAGNA I (c. 1525 A.D.)	Anantasudhārasa	
47	ANANTADAIVAGNA II (c. 1537 A.D.)	Kāmadhenuganita-tīka	
48	SANKARA alias MAZHAMANGALATH NAMBUDIRI (c. 1533 A.D.)	(i) Muhūrthapadavi (1544 A.D.) (ii) Kāladīpakam	
		(iii) Laghubhāskarīyam	A Malayālam translation of Bhāskarā's work
		(iv) Ganita-sāra	An elementary work in Malayālam verse
		(v) Çandraganitakrama (vi) Prasnasara (vii) Pançabōdha darpanam (viii) Pançabōdham	(v)—(vii) are again in Malayālam

Commentator's name and date	Name, date and general nature of the commentary	Remarks
		The com. has been printed in Leiden
		Edited recently by Dr. A. N. Singh
		Printed at Ānandāsram, Poona.
PUNNĀGA KRŌDA BHIKSHUVARYA	A commentary in Malayālam	Text printed in T.S.S.; the commentary is available with H. H. the 10th Prince of Cochin A MS. available in the Skt. College Library, Cochin
		An incomplete MS. available with me
VISVANĀTHA		MS. in Benares Govt. Library
(i) DIVĀKARA (ii) VISVANĀTHA (iii) GOKULANĀTHA	(i) Vivaraṇa (ii) Udāharana	
		Jātakapaddhati and Jātakapaddhati-vivrti are the author's astrological works
		Published in T.S.S.
	A lucid commentary in Sanskrit	MS. in Skt. College Library, Trippunithura (Cochin) MS. in Adayār Library, Madras
		MS. available in Govt. Library, Benares
MALLĀRI (*c.* 1571 A.D.)		The author has written a number of astrological works besides
		Printed in Trivandrum
		All these works are collectively referred to as 'Bālaśānkaram' Printed at Mangalodyam Press, Trichur

Sl. No.	Author's name, date, etc.	Name of the work	General nature of the work
49	VISHNU (*c*. 1556 A.D.)	Vishnukaranam	A work of the Soura School
50	SŪRYADĀSA (*c*. 1541 A.D.; disciple of Gnānarāja)	(i) Gaṇitakūpika (ii) Bījaprakōsa	A commentary on Līlâvati Notes on Bhāskara's Bījagaṇita
51	NĪLAKANTA DAIVAGNA (*c*. 1575 A.D.; father of Ananta II, and one of Akbar's courtiers)	Nīlakanti	
52	KṚṢNADAIVAGNA (*c*. 1565 A.D., a courtier of Jehangir)	Navānkura	An original commentary on Bījaganita
53	RAGHUNĀTHA (*c*. 1565 A.D.)	Manipradīpam	A treatise following the Sūrya siddhānta
54	RĀMADAIVAGNA (*c*. 1565 A.D ; son of Ānanda II, brother of Nīlakanta)	(i) Râmakarana (ii) *Yantraprakāsa* (iii) Muhūrthaçintâmani	Composed at the instance of the Raja of Jaipur
55	GOVINDA DAIVAGNA (*c*. 1569 A.D.; son of Nīlakanta)	Pīyūshadhārā	A gloss on Muhūrtaçintamaṇi
56	VISVANĀTHA (*c*. 1728)		
57	NṚSIṂHA (1586 A.D.)		
58	NĀRĀYANA (*c*. 1588 A.D., 'guru' of Munīsvara)	Nārâyanībîja	
59	SIVA (*c*. 1591 A.D.)	Anantasudhâvivrti	Commentary on Anantasudhārasa
60	SRĪNĀTHA (*c*. 1592 A.D.		
61	SŌMADAIVAGNA (*c*. 1602 A D.)	Kalpalata	Deals with the Pançāngas
62	MUNĪṢVARA (*c*. 1603 A.D.)	(i) Siddhānta-sārvabhauma (ii) Līlâvatītika (iii) Spashtīkaranabhangi	A superb commentary on Siddhānta-sirōmani in 3000 verses
63	DIVĀKARA (*c*. 1603 A.D.)	Makarandapaddhati	
64	KAMALĀKARA (*c*. 1613 A D.)	Siddhānta tatva vivéka	This belongs to the Saura School and attacks Bhāskara and Munīsvara
65	SANKARA (*c*. 1607 A.D.)	Vaishnavakaranam	
66	JATĀDHARA (*c* 1614 A.D.)	? Phathesāha prakāsa	
67	NĀGÉSA (c. 1618 A.D.)	Grahaprabōdha	
68	BRAHMADATTA (*c*. 1639 A.D.)	Yuktibhāshā	Presumably a Malayālam translation of Gaṅitayukti (Madras Library)
69	NITYĀNANDA (*c*. 1639 A D.)	Siddhantarāja	A book revealing sure Persian influence
70	RANGANĀTHA (*c*. 1643 A D.)	(i) Siddhāntaçūdâmani (ii) Laghubhangīvibhāngi	An excellent work in 400 stanzas An attack on Munīsvara
71	JAGANNĀTHA (*c*. 1652 A.D. A Telugu Brāhmin pandit of the Court of Raja Jai Singh)	(i) Siddhāntasamrāt (ii) Rékhāganita	A translation of Euclid undertaken on the order of the King
72	KṚṢNA (*c*. 1653 A.D.)	Karanakausthubha	
73	RATNAKĀNTA (*c*. 1658 A.D.)	Pançāngakautuka	
74		Grahatarangini (1696 A.D.)	
75		Siddhântamanjari (1719 A.D.)	

Commentator's name and date	Name, date and general nature of the commentary	Remarks
VISVANĀTHA (*c.* 1623 A.D.)	The author's brother	A versatile author with works in many other fields to his credit
VISVANĀTHA	Udāharana	
		MS in Benares
VISVANĀTHA		MS. available in Benares
VISVANĀTHA		
		A very prolific commentator
		An incomplete MS. available in Benares
		MS. available with me. Published since by Mangalodayam Press, Trichur
		Both these can be had in Benares
		An Ānandāshram edition exists

Sl. No.	Author's name, date, etc.	Name of the work	General nature of the work
76		Pançāngasirōmani	
77	ÇINTĀMANI DĪKSHITA (*c.* 1736 A.D.)	(1736 A.D.)	
78	MANĪRĀMA (*c.* 1774 A.D.)	Grahagaṇita çintāmani	
79		Brahmasiddhānta-sāra (1781 A.D.)	
80	MATHURĀNĀTHA (*c.* 1782 A.D.)	Yantrarajaghatanā	
81	RĀGHAVA (*c.* 1810 A.D.)		
82	SIVA (*c* 1814 A.D.)	Tithipārıjāta	
83	SANKARA VARMA (*c.* 1820 A D.)	Sadratnamāla	
84	BĀPUḌĖVA SĀSTRY		
85	SUDHĀKARA DVIVĖDI		
86	PURUSHOTTAMA SARMA (1940 A.D.)	Ganitanirna a	An amplification of the Dṛg System

Commentator's name and date	Name, date and general nature of the commentary	Remarks
		MS. available at the Adayar Library
		Prominent modern editors of recent times
		An astrologer of repute besides

work; in several others they have been arrived at by a judicious sifting of evidence, both internal and external. In general, however, it is advisable to demur from accepting these latter as final. In some instances no clue could be obtained to determine the period.

Every first-rate work has inspired first-rate commentaries which sometimes even outshine their originals. The commentators were in several instances the direct pupils of their authors and had independent works to their credit.

All but the careless skipper of the following pages will notice two outstanding imperfections in this list. First, the treatment is more complete and thorough at the beginning than at the end; this is because the later works are in the main commentaries (sometimes very original doubtless) and so do not deserve as detailed a scrutiny as the classics. Secondly, a certain amount of overlapping and redundance has crept in, but this was unavoidable.

The indebtedness of the compiler to the following sources (over and above the huge volume of periodical literature on the subject) can hardly be overestimated:

1. Gaṇaka Tarangini by Sudhākara Dvivédi.
2. Mahābhāskarīya—Introduction in the Ānandāsram edition.
3. History of Sanskrit Literature in Malabar by Rāja Rāja Varma.
4. The compiler's own collection of MSS.

THE BICIRCULAR GENERATION OF A CONIC

BY

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1. The use of directed lines and circles in elementary geometry and trigonometry is well known, and can be recalled to mind by a few examples.

If two lines are directed, the locus of the centre of a variable circle which they both touch is one line, not a pair of lines; thus we have an intrinsic difference between the two angle-bisectors. There is another difference: only one of the angle-bisectors is an axis of symmetry, for reflection in the other angle-bisector gives lines with the wrong directions.

Three directed lines have only one contact-circle, not four. Two directed circles have only one centre of similarity, and have at most one pair of common tangents. The three centres of similarity of three directed circles taken two at a time are collinear, and three directed circles may have two contact-circles or none, but cannot have more than two.

A set of three directed circles is anallagmatic with respect to their radical centre: inversion in this point, with the appropriate constant, followed by a reversal of directions if this constant is positive, reproduces the original set. Since this operation transforms one contact-circle into another contact-circle, it follows that if three directed circles have one contact-centre, they have two contact-centres, and these two points are in line with the radical centre.

If three undirected circles have in fact eight contact-circles, the eight ways of assigning directions to the three circles give eight distinct pairs of contact-circles, but since a simultaneous reversal of all the directions in the figure does not disturb any contacts that exist, each of these pairs is the reversal of one other pair; the eight contact-centres of the undirected set fall into pairs on four lines through the radical centre.

In all the examples the principle is the same. The imposition of direction eliminates some of the possibilities from the undirected figure, and the complexities of the figure of undirected lines and circles are restored when several simpler figures are superposed. The object of this paper is to indicate an application of the same principle to a group of theorems on conics.

2. If P is a variable point on a conic whose foci are A, B, the image of A in the tangent at P is a point H on the line BP, and since the midpoint of AH describes a circle round the midpoint of AB, the point H describes a circle round B. Hence P is the centre of a circle which passes through the

fixed point A and touches a fixed circle round B; the circle being regarded as variable, the conic is the locus of its centre. This property is equivalent to the bifocal property, but without the ambiguity of sum or difference.

It is easily seen that this mode of generating the conic must admit of some kind of extension. If the conic is an ellipse, any point P on the conic is inside the fixed circle. If the circle with centre P and radius r, or as we shall call it the circle (P, r), passes through A and touches the circle (B, h), then, for any positive value of k, the circle $(P, r+k)$ touches the two fixed circles (A, k) and $(B, h+k)$. That is, the ellipse can be generated bicircularly, and there is an arbitrary parameter in the specification of the circles on which it depends. If k is to be subtracted, or if the conic is not assumed to be an ellipse, the argument needs modification, and even in the case with which we have dealt there is a complication, for it is not every circle which touches both (A, k) and $(B, h+k)$ whose centre is on the original ellipse. On the contrary, the locus of the contact-centres of the two circles includes, as we proceed to shew, a second ellipse.

Consider more generally two circles (A, g), (B, h), and a point P whose distances from A, B are r_A, r_B. A circle round P touches (A, g) if its radius is either the sum $r_A + g$ or the difference $r_A \sim g$, and touches (B, h) if its radius is either the sum $r_B + h$ or the difference $r_B \sim h$. Hence P is a contact-centre of the two fixed circles if either the sum $r_A + r_B$ or the difference $r_A \sim r_B$ has one of the two constant values $g + h$, $g \sim h$, that is, if P is on a conic with foci A, B whose transverse axis measures either $g + h$ or $g \sim h$. It is only if g or h is zero, that is, only in the case with which we began, that the two possibilities coincide and the locus of P reduces to a single conic, though we may notice that if g and h are equal, or if their sum or difference is equal to the distance between A and B, one part of the locus degenerates and is easily ignored. In general the locus of P is a pair of conics, and we cannot utilise a bicircular generation of a conic in which we are interested unless we are able to eliminate an unwanted companion.

3. Again we have recourse to the device of giving directions to circles, and we suppose that the radii g, h of the circles round A, B have significant signs. The circle (P, r) touches (A, g) if $|r - g| = |AP|$, that is, if $r = g + r_A$, and touches (B, h) if $|r-h| = |BP|$, that is, if $r = h + r_B$. When the radius r is eliminated, only the difference $g-h$ between the fixed radii can be involved in the result, and P is a contact-centre if either $r_A + r_B$ or $r_A \sim r_B$ has the value $|g-h|$. If $|g-h| > |AB|$, the difference $r_A \sim r_B$ cannot be equal to $|g-h|$, and the locus of P is the ellipse $r_A + r_B = |g-h|$; if $|g-h| < |AB|$, the sum $r_A + r_B$ cannot be equal to $|g-h|$ and the locus is the hyperbola $r_A \sim r_B = |g-h|$. In fact the locus is the one conic with A and B for foci whose transverse axis is of length

$|g-h|$, and this same conic is determined by the circles $(A, g+k)$, $(B, h+k)$ for any value of k, positive or negative. The conic is determined alternatively by the circles $(A, h+k)$, $(B, g+k)$, but these circles may be regarded rather as obtained by reversing the circles $[A, g-(g+h+k)]$, $[B, h-(g+h+k)]$ than as belonging to a distinct family of generating circles.

The focus-directrix property is deducible immediately from the bicircular generation. If (P, r) touches (A, g), the power π_α of P for (A, g) is $r(r-2g)$, and if the same circle round P touches (B, h), the power π_β of P for this last circle is $r(r-2h)$. But if PL is the perpendicular from P on the radical axis of the circles round A and B, then $\pi_\alpha - \pi_\beta = 2AB.LP$, unambiguously since LP is parallel to AB. Hence if a definite direction of measurement is adopted for distances parallel to the line of centres AB, then $r = e.LP$ where $e = AB/(h-g)$, and further $r-g = e.MP$, $r-h = e.NP$ where M, N are the points on LP such that $e.LM = g$, $e.LN = h$. That is, $r_A = |e|.|MP|$, $r_B = |e|.|NP|$, where PM, PN are the perpendiculars from P on the two fixed lines at distances g/e, h/e from the radical axis.

4. There is little satisfaction in using a particular method of generating a conic only to recover familiar properties, and we apply the bicircular generation at once to a problem for which it is peculiarly suitable. Let α, β, γ be a focus-sharing triad of conics, α having B and C for foci, β having C and A for foci, and γ having A and B for foci. We may suppose that in the first place α is determined by the point B and a circle (C, f), β by the point C and a circle (A, g), and γ by the point A and a circle (B, h). Then equally α is determined by the pair of circles (B, y), $(C, f+y)$ where y is arbitrary, β by the pair of circles (C, z), $(A, g+z)$ where z is arbitrary, and γ by the pair of circles (A, x), $(B, h+x)$ where x is arbitrary. Since the set of equations

$$f+y = -z, \quad g+z = -x, \quad h+x = -y$$

is always consistent, we can require the two circles round each focus to differ only in direction, that is, analytically, only in the sign of the radius. Thus we obtain what we call a tricircular basis for the focus-sharing triad.

We write

$$x_0 = k-f, \; y_0 = k-g, \; z_0 = k-h,$$

where $2k = f+g+h$, and we denote the circles (A, x_0), (B, y_0), (C, z_0) by ξ_0, η_0, ζ_0 and the reverse circles $(A, -x_0)$, $(B, -y_0)$, $(C, -z_0)$ by $\xi_0', \eta_0', \zeta_0'$; the undirected circles with centres A, B, C and radii x_0, y_0, z_0 can be denoted by $|\xi_0|$, $|\eta_0|$, $|\zeta_0|$. With this notation, there are bicircular generations of α from η_0 and ζ_0', of β from ζ_0 and ξ_0' and of γ from ξ_0 and η_0'. We can generate γ as well from ξ_0' and η_0 as from ξ_0 and η_0', and then we are using the same circle round A for the two conics which have A for a focus, but,

since we cannot generate α from η_0 and ζ_0, at least one reversal is essential and we gain nothing by sacrificing symmetry of treatment.

If there is a circle which touches the three circles ξ_0', η_0, ζ_0, the centre of this circle is a point common to β and γ, and since there is in this case a second contact-circle, the two conics have a chord of intersection associated with the triad of circles. This chord passes through the radical centre of the circles, a point R_0 which belongs to the tricircular basis as a whole, since it is independent of the directions assigned to the circles. If the three undirected circles $|\xi_0|$, $|\eta_0|$, $|\zeta_0|$ possess the full complement of eight contact-circles, each of the three sets $\xi_0'\eta_0\zeta_0$, $\xi_0\eta_0'\zeta_0$, $\xi_0\eta_0\zeta_0'$ has its own pair, and then

Associated with a tricircular basis of a focus-sharing triad of conics α, β, γ there is a triad of chords of intersection, one through two points common to β and γ, one through two points common to γ and α, and one through two points common to α and β, concurrent in the radical centre of the basis.

5. It is clear that the analysis in the last paragraph is incomplete. The set of circles $\xi_0'\eta_0\zeta_0$ has at most two contact-centres, but the two conics β, γ may cut in four points and may have six chords of intersection. The explanation is simple. There are two circles round a given point P which touch the circle ξ_0', and all that is necessary to characterize P as a point common to β and γ is that one of these circles should touch η_0 and one of them should touch ζ_0; we have considered the case in which the same circle touches both η_0 and ζ_0, and there remains the case in which one touches η_0 and the other touches ζ_0. When this second case occurs the conics have a second chord of intersection, but we have no reason to expect this chord to pass through the radical centre R_0.

We can avoid direct examination of the more awkward alternative. The tricircular basis $\xi_0\eta_0\zeta_0$ has been defined uniquely in terms of the original radii f, g, h, and the conics α, β, γ themselves are altogether independent of the signs of these radii. Changing the signs of f, g, h in turn we have three other bases: $\xi_1\eta_1\zeta_1$ with radii k, $h-k$, $g-k$; $\xi_2\eta_2\zeta_2$ with radii $h-k$, k, $f-k$; $\xi_3\eta_3\zeta_3$ with radii $g-k$, $f-k$, k. A further change of sign only reverses completely the directions in a basis already found, but the four bases specified are essentially distinct, and we have four radical centres R_0, R_1, R_2, R_3, with the possibility of four sets of concurrent chords.

If the circle (P, r) touches the three circles $(A, f-k)$, $(B, k-g)$, $(C, k-h)$, then the circle $(P, r-f)$ touches the three circles $(A, -k)$, $(B, h-k)$, $(C, g-k)$; that is to say, a point which is the centre of a circle touching ξ_0', η_0, ζ_0 is also the centre of a circle touching ξ_1', η_1, ζ_1. Hence the basis $\xi_1\eta_1\zeta_1$ picks out the points of intersection of β and γ in the same way as the basis $\xi_0\eta_0\zeta_0$; if there are contact-centres, we find them again, and we learn

nothing about common points which are not contact-centres. Nevertheless we have learnt more about the contact-centres, for these are in line with R_1 as well as with R_0, and therefore, since R_1 is not identical with R_0, the line associated with each of the tricircular bases $\xi_0\eta_0\zeta_0$, $\xi_1\eta_1\zeta_1$ as a chord of intersection of the two conics β, γ is the line R_0R_1.

To change the sign of f transforms the basis $\xi_2\eta_2\zeta_2$ into $\xi_3'\eta_3'\zeta_3'$; it follows that if $\xi_2'\eta_2\zeta_2$ or $\xi_3'\eta_3\zeta_3$ had the same contact-centres as $\xi_0'\eta_0\zeta_0$, the four radical centres would be collinear on a chord of intersection of β and γ, and this must be exceptional. We are prepared therefore to find that a contact-centre of $\xi_2'\eta_2\zeta_2$ has the property of a point common to β and γ which is not a contact-centre of $\xi_0'\eta_0\zeta_0$. Verification is simple. Let (P, r) touch $(A, k-h)$, (B, k), $(C, f-k)$; then $(P, r-g)$ touches $(A, f-k)$ and $(B, k-g)$, that is, ξ_0' and η_0, while since $(P, -r)$ touches $(A, h-k)$ and $(C, k-f)$, it follows that $(P, -r-h+f)$ touches $(A, f-k)$ and $(C, k-h)$, that is, ξ_0' and ζ_0; also, since the sum of the radii $r-g$, $-r-h+f$ is $2(f-k)$, the circles $(P, r-g)$, $(P, -r-h+f)$ are the two distinct circles round P which touch ξ_0'. The points common to β and γ which are not contact-centres of $\xi_0'\eta_0\zeta_0$ and $\xi_1'\eta_1\zeta_1$ are therefore the contact-centres of $\xi_2'\eta_2\xi_2$ and of $\xi_3'\eta_3\zeta_3$, and the bases $\xi_2\eta_2\zeta_2$, $\xi_3\eta_3\zeta_3$ associate the line R_2R_3 with the two conics β, γ as a second chord of intersection.

The radical centres of the four tricircular bases from which a focus-sharing triad of conics can be generated form a quadrangle whose pairs of sides are pairs of chords of intersection of the conics taken two by two.

In our notation, the lines R_0R_1 and R_2R_3 are chords of intersection of β and γ, the lines R_0R_2 and R_3R_1 are chords of intersection of γ and α, and the lines R_0R_3 and R_1R_2 are chords of intersection of α and β. The lines, it will be noticed, are real lines, whether or not the conics really cut them.

6. If e_β, e_γ are the eccentricities $c/(x_0+y_0)$, $b/(z_0+x_0)$ of the conics β, γ, we have seen in § 3 that the distance of the directrix of β associated with A from the radical axis of $|\xi_0|$ and $|\eta_0|$ is $-x_0/e_\beta$ and the distance of the directrix of γ associated with A from the radical axis of $|\zeta_0|$ and $|\xi_0|$ is x_0/e_γ. Similarly the distances of the same directrices from the radical axes of $|\xi_1|$ and $|\eta_1|$ and of $|\zeta_1|$ and $|\xi_1|$ are $-x_1/e_\beta$ and x_1/e_γ. Hence the ratio of the distances of R_0 and R_1 from the directrix of β at A is the same as the ratio of the distances of these points from the directrix of γ at A, and therefore the line R_0R_1 passes through the intersection of the two directrices. The proof applies equally well to the line R_2R_3. Thus

The quadrangle formed by the radical centres of a focus-sharing triad has for its diagonal points the intersections of the pairs of directrices at the three foci.

It is evident on the most elementary grounds that if two conics have a common focus, the line joining the point of intersection of the two directrices at that focus to one point of intersection of the conics passes through a second point of intersection of the conics. Thus the possession of a common focus identifies one of the diagonal points of the quadrangle of points of intersection, and selects a particular pair of common chords. Each pair of conics in a focus-sharing triad has one pair of common chords selected in this way, and the theorem of this paragraph proves that, as we should expect, it is the three pairs concurrent with pairs of directrices that are the pairs of sides of the radical quadrangle.

7. There remains a question which is not without interest. A line which plays the part of a chord of intersection of two conics may or may not cut the curves, as we may say, visibly. If in a focus-sharing triad some of the sides of the radical quadrangle are chords intersecting the curves visibly and others are not, can we make any assertion as to concurrence among chords of visible intersection?

We choose f, g, h to be positive, and we choose the directions of measurement parallel to the focal axes to be such that BC, CA, AB are all positive; then also the three eccentricities e_α, e_β, e_γ are positive. We denote the intersections of the pair of directrices at A by F, of the pair of directrices at B by G, and of the pair of directrices at C by H. Lastly, we denote the distances of a variable point of the plane from the parallel directrices of α by u_G and u_H, with corresponding notation for distances from the directrices of β and γ.

The equation of the line R_0R_1F has the form $e_\beta v + e_\gamma w = 0$ if the axes are any lines perpendicular to CA and AB and the origin is any point on the line itself, and the equation of R_2R_3F is $e_\beta v - e_\gamma w = 0$ if the axes are perpendicular to CA, AB and the origin is any point on R_2R_3F. In particular, taking the origin at F, every point on R_0R_1 satisfies the equation $e_\beta v_F + e_\gamma w_F = 0$, and every point on R_2R_3 satisfies the equation $e_\beta v_F = e_\gamma w_F$.

Suppose first that the conic β is an ellipse; for every point of β, v_H is positive and v_F is negative. If γ is an ellipse, then for every point of γ, w_F is positive and w_G is negative; it is then impossible for $e_B v_F$ to be equal to $e_\gamma w_F$ at any point common to β and γ, and therefore if β and γ intersect visibly, their chord of visible intersection is R_0R_1. If γ is a hyperbola, then w_F is negative on the branch of γ nearer to A and positive on the branch further from A; in this case β can intersect the nearer branch on R_2R_3 and the further branch on R_0R_1, but since it is impossible for an ellipse which surrounds A to cut the farther branch without cutting the nearer branch also, it follows that if β and γ have only one chord of visible intersection, this chord is R_2R_3.

If β is a hyperbola and γ an ellipse, the conclusion is the same: there may be no common points, or there may be four, but if there are precisely two, these two are on R_2R_3.

If both β and γ are hyperbolas, the argument is more elaborate. Since β has a near branch on which v_F is positive and a far branch on which v_F is negative, and γ has a near branch on which w_F is negative and a far branch on which w_F is positive, a common point on R_2R_3 is a point where the near branch of one conic cuts the far branch of the other. Suppose that P is a point where the near branch of β cuts the far branch of γ. Then the line AP cuts the near branch of γ in a point X which is between A and P, and cuts the far branch of β in a point Y which is inside the far branch of γ. Since the axes of β and γ are not parallel, at least one of the asymptotes of γ is not parallel to either asymptote of β; draw a line through X which is parallel to an asymptote of γ and not of β; every point of this line is outside the far branch of γ. Since X is inside the near branch of β, this line cuts β in two points U, V of which one is certainly on the near branch. If U and V are both on the near branch of β, then X is between them, one is inside and one is outside the near branch of γ, and therefore the near branches of the two curves intersect. If on the other hand U is on the near branch of β and V is on the far branch, then the far branch of β includes a point Y inside the far branch of γ and a point V outside the far branch of γ, and therefore the far branches of the two curves intersect. Thus if β and γ cut visibly on R_2R_3, they cut visibly on R_0R_1 also.

To sum up, if β and γ have only one chord of visible intersection, this chord is R_0R_1 if the two conics are of the same kind, and is R_2R_3 if one of them is an ellipse and the other is a hyperbola. Hence if each pair in the triad α, β, γ has one and only one chord of visible intersection, the chords are the three chords concurrent in R_0 if the conics are all of the same kind, but if α is of one kind and β and γ are of the other kind, they are the three chords concurrent in R_1. In any case,

If each pair of conics in a focus-sharing triad has one and only one chord of visible intersection, the three chords are concurrent.

8. The study of focus-sharing triads was begun in 1913 by J. S. Turner, who drew a large number of sets of focus-sharing ellipses and found that if each pair of ellipses in a set possessed a chord of intersection, then however he varied the shape of the focal triangle or the lengths of the major axes, the three chords belonging to the set were always concurrent within the limits of draughtsmanship. In due course, with the help of J. L. Coolidge, Turner convinced himself algebraically that he had stumbled on a true proposition, and he announced the theorem without proof in 1929. In 1936 the theorem that a focus-sharing triad provides three pairs of chords of inter-

section which are the pairs of sides of a quadrangle, proved by a transformation of tripolar co-ordinates, was communicated to the *Quarterly Journal of Mathematics*, in a paper published more than two years later.* Almost simultaneously a short paper containing direct proofs of this theorem and on the theorem of §7 above, which includes Turner's result, was sent to the *Mathematical Gazette*, where publication was prompt.† The direct proofs are elementary, but if they shewed us that the theorems are not profound, they did nothing to lessen the unexpectedness of the conclusions. With the introduction of the tricircular basis, not even a puzzle remains.

* *Multipolar and Multiglobular Co-ordinates*, Dec. 1938, 9, 294. The theory of the co-ordinates was reduced to its simplest form for a paper in the *J.I.M.S.* entitled "Bipolar and trigeminal co-ordinates on a line," 1937, 2, 173.

† *A focus-sharing set of three conics*, July 1936, 20, 182.

ON THE APPROXIMATION OF A FUNCTION BY A POWER SERIES

BY

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1. This note contains a study concerning the structure of the differential calculus. It investigates the approximation of a function in the neighbourhood of a value a through a power-series, *i.e.*,

$$f(a+h) = f(a) + ha_1 + h^2a_2 + \dots + h^n\left(a_n + \epsilon(h)\right)$$

with $\lim_{h \to 0} \epsilon(h) = 0$. We shall investigate conditions for the existence of such an approximation for a certain value of n and give a procedure for finding the coefficients a_k. For $n = 1$ the existence of this approximation is identical with the existence of the derivative $f'(a)$ of f at the point a. We write:

$$Q_1(h) = \frac{f(a+h) - f(a)}{h}, \quad D_1 = \lim_{h \to 0} Q_1(h)$$

$$Q_k(h) = \frac{Q_{k-1}(h) - D_{k-1}}{h}, \quad D_k = \lim_{h \to 0} Q_k(h),$$

then the existence of the approximation for an arbitrary n is identical with the existence of the successive limits $D_1, D_2, \dots, D_n$, and it is $D_k = a_k$. The limiting process immediately gives the coefficients a_k. If the nth derivative (and therefore each derivative of lower order) exists and is continuous at a, then D_n is equal to $n\,!\,f^{(n)}(a)$.

2. The theory of approximation consists, now, in finding other sufficient or even necessary and sufficient conditions for the existence of such an approximation. The Taylor theorem says that, if the nth derivative exists and is continuous in an interval around a, then the approximation exists, with $\epsilon(h)$ in different forms This condition is not at all necessary. The proof that it is sufficient involves the use of mean value theorems or the fundamental theorems of the integral calculus, all of them difficult from the systematic point of view. These complications have to be accepted as a price for the highly important connection with the derivatives and for the advantage that the derivatives are easy to compute for the usual functions.

In the following discussion we establish a set of necessary and sufficient conditions which seem to have the advantage of involving essentially simpler limiting processes than the immediate condition that D_k $(k = 1, \dots, n)$ exists. Looking at the definition of D_k we see that the D_k are defined by an iterated

limit process whereas the limit processes in the new conditions are independent from each other. Furthermore these limiting processes are, if not as easily performed as those leading to the derivatives, in the most important cases to be performed by algebraic computations. Thus one finds the expansions of the exponential function and the trigonometric functions without using higher derivatives.

3. We define

$$(3.1)\quad R_1(h) = \frac{f(a+h) - f(a)}{h}$$

$$(3.2)\quad R_k(h) = \frac{R_{k-1}(2h) - R_{k-1}(h)}{h}.$$

Then we shall prove that *the existence of*

$$S_k = \lim_{h \to 0} R_k(h)$$

for $k = 1, 2, \ldots, n$ *is necessary and sufficient for the existence of the approximation*

$$f(a+h) = f(a) + ha_1 + h^2 a_2 + \ldots + h^n \left(a_n + \epsilon(h)\right)$$

with $\lim\limits_{h \to 0} \epsilon(h) = 0$.

4. From the existence of the approximation, it follows that

$$R_1(h) = a_1 + ha_2 + h^2 a_3 + \ldots + h^{n-1}\left(a_n + \epsilon(h)\right)$$

$$S_1 = a_1.$$

If now for one index $k = 1, 2, \ldots, n-1$

$$(4)\quad R_k(h) = \sum_{l=0}^{n-k} (2^{l+1} - 1)(2^{l+2} - 1)\ldots(2^{l+k-1} - 1)\, h^l a_{l+k} + h^{n-k} \epsilon_k(h)$$

with $\lim\limits_{h \to 0} \epsilon_k(h) = 0$, where the $k-1$ numerical factors of $h^l{}_{l+k}$ are to be omitted in the case $k = 1$, then the formula (3.2) implies

$$R_{k+1}(h) = \sum_{l=1}^{n-k} (2^l - 1)(2^{l+1} - 1)\ldots(2^{l+k} - 1)\, h^{l-1} a_{l+k} + h^{n-k-1} \epsilon_{k+1}(h)$$

with $\lim\limits_{h \to 0} \epsilon_{k+1}(h) = 0$. Consequently, (4) holds for $k = 1, 2, \ldots, n$, and we conclude

$$S_k = \lim_{h \to 0} R_k(h) = 1.\ 3.\ 7.\ \ldots,\ (2^{k-1} - 1)\, a_k,\quad k = 1, 2, \ldots, n.$$

5. In this simple way we have proved that for the existence of the approximation the existence of the limits S_k is *necessary*. The proof that the conditions are *sufficient* is not quite as simple.

If we assume that S_n exists we have:

$$(A)\quad R_{n-1}(2h) = R_{n-1}(h) + h\,\epsilon_n(h) + h\,S_n, \text{ with } \lim_{h \to 0} \epsilon_n(h) = 0.$$

We apply this relation for $R_{n-1}(h)$, substitute this value of R_{n-1} in (A) and have:

$$R_{n-1}(2h) = R_{n-1}\left(\frac{h}{2}\right) + h(1+\tfrac{1}{2})\, S_n + h\left(\epsilon_n(h) + \frac{\epsilon_n\left(\frac{h}{2}\right)}{2}\right).$$

Repeating this process we have:

$$R_{n-1}(2h) = R_{n-1}\left(\frac{h}{4}\right) + h(1+\tfrac{1}{2}+\tfrac{1}{4})\, S_n + h\left(\epsilon_n(h) + \frac{\epsilon_n\left(\frac{h}{2}\right)}{2} + \frac{\epsilon_n\left(\frac{h}{4}\right)}{4}\right).$$

We continue in this wav and get: $R_{n-1}(h) =$

$$R_{n-1}\left(\frac{h}{2^m}\right) + h\left(1+\tfrac{1}{2}+\tfrac{1}{4}+\dots\frac{1}{2^m}\right) S_n + h\left(\epsilon_n(h) + \frac{\epsilon_n\left(\frac{h}{2}\right)}{2} + \dots \frac{\epsilon_n\left(\frac{h}{2^m}\right)}{2^m}\right).$$

If m goes to infinity the factor of S_n goes to $2h$, and according to our assumption for R_{n-1}, $R_{n-1}\left(\frac{h}{2^m}\right)$ goes to S_{n-1}, finally, because of the assumption for R_n, the ϵ_n- series is convergent and its sum goes to *zero* with h. We have therefore:

(B) $\quad R_{n-1}(2h) = S_{n-1} + 2hS_n + 2h\epsilon_{n-1}(h)$ with $\lim\limits_{h\to 0} \epsilon_{n-1}(h) = 0.$

By definition

$$R_{n-1}(2h) = \frac{R_{n-2}(4h) - R_{n-2}(2h)}{2h}.$$

Substituting (B) in this expression we have:

$$R_{n-2}(4h) = R_{n-2}(2h) + 2hS_{n-1} + 4h^2 S_n + 4h^2\epsilon_{n-1}(h).$$

Using this relation for $R_{n-2}(2h)$, for $R_{n-2}(h)$ and so on, taking the assumption for R_{n-2} and summing the infinite geometric progression $1 + \frac{1}{4} + \frac{1}{4^2} + \dots$, we obtain:

(C) $\quad R_{n-2}(4h) = S_{n-2} + 4h\, S_{n-1} + \left(\frac{(4h)^2}{3}\right) S_n + (4h)^2\, \epsilon_{n-2}(h)$

with $\lim\limits_{h\to 0} \epsilon_{n-2}(h) = 0.$

If we take for example $n = 3$, we have:

$$R_1(4h) \equiv \frac{f(a+4h) - f(a)}{4h} = S_1 + 4hS_2 + \frac{(4h)^2}{3} S_3 + (4h)^2\, \epsilon_1(h)$$

$$\frac{f(a+h) - f(a)}{h} = S_1 + h\, S_2 + \frac{h^2}{3} S_3 + h^2\, \epsilon_1\left(\frac{h}{4}\right)$$

or

$$f(a+h) = f(a) + hS_1 + h^2S_2 + \frac{h^3}{3} S_3 + h^3\, \epsilon_1\left(\frac{h}{4}\right)$$

with $\lim\limits_{h\to 0} \epsilon_1(h) = 0.$

For general n we have, using the assumptions for $R_n, R_{n-1}, \ldots R_1$ and summing the infinite geometric progressions $1+\frac{1}{2^m}+\frac{1}{2^{2m}}+\ldots$:

$$f(a+h)=f(a)+hS_1+\frac{h^2}{1}S_2+\frac{h^3}{1.3}S_3+\ldots\frac{h^n}{1.3..(2^{n-1}-1)}(S_n+\epsilon(h))$$

with $\lim_{h\to 0}\epsilon(h)=0$ in conformity with the results in (4). Thus it is proved that the existence of the S_k is sufficient for the existence of the approximation.

6. As an example we compute the approximation for e^h:

$$R_1(h)=\frac{e^h-1}{h},\quad S_1=1$$

$$R_2(h)=\frac{\frac{e^{2h}-1}{2h}-\frac{e^h-1}{h}}{h}=\frac{(e^h-1)^2}{2h^2},\quad S_2=\tfrac{1}{2}$$

$$R_3(h)=\frac{(e^h-1)^3(e^h+3)}{8h^3},\quad S_3=\tfrac{1}{2}$$

$$e^h=1+h+\frac{h^2}{2}+\frac{h^3}{3}\left(\tfrac{1}{2}+\epsilon(h)\right).$$

The computation of higher terms in the approximation involves the recurrent computation of the polynoms $P_n(x)$ where

$$P_n(x)=\frac{(x+1)^{n-1}P_{n-1}(x^2)-2^{n-1}P_{n-1}(x)}{x-1}.$$

7. A disadvantage in using our limits S_k consists in the arbitrariness connected with the comparison of the quotients for $2h$ and for h. We could as well have taken rh $(r>1)$ instead of $2h$. The existence of the corresponding limits is equivalent to existence of the S_k.

The existence of the S_k implies the existence of the limits of the higher difference quotients. For instance, from the existence of S_1, S_2, S_3 follows the existence of

$$\lim_{h\to 0}\frac{f(a+3h)-3f(a+2h)+3f(a+h)-f(a)}{h^3}.$$

It seems however that the existence of these limits does not imply the existence of the S_k and the existence of the approximation. It would be interesting to prove this conjecture by an example.

8. It is obvious that using the limits S_k we can express curvature, torsion, etc. without assuming the existence of higher derivatives.

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ON EQUAL SUMS OF POWERS

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In this paper, I give certain general, though not perhaps complete solutions of simultaneous identities, which express the equality of sums of more than five powers.

1. (1) Consider the relation $7G^2 = H^2 + HJ + J^2$, for which Mahrenholz has given the solution

$$H = 2a^2 - 2ab - 3b^2; \; J = a^2 + 6ab + 2b^2; \; G = a^2 + ab + b^2.$$

Then we have

$$7G + H,\; 3H - G,\; 2J - H + G,\; 3H + 2J - G \stackrel{n}{=} 7G - H,\; 3H + G,$$
$$2J - H - G,\; 3H + 2J + G,$$

identically true for $n = 2, 4, 6$, and we write the solutions as $A_1, A_2, A_3, A_4 \stackrel{n}{=} B_1, B_2, B_3, B_4$ for $n = 2, 4, 6$.

Example.—$a = 2$, $b = 1$ gives $G = 7$, $H = 1$, $J = 18$, and after dividing the expressions obtained by 2, we have the result

$$25,\; -2,\; 21,\; 16 \stackrel{n}{=} 24,\; 5,\; 14,\; 23$$

for $n = 2, 4, 6$.

Given $\sum_{i=1}^{4} A_i^n = \sum_{i=1}^{4} B_i^n$ for $n = 2, 4, 6$, we have for arbitrary t the simultaneous identity

$$t - A_4,\; t - A_3,\; t - A_2,\; t - A_1,\; t + A_1,\; t + A_2,\; t + A_3,\; t + A_4 \stackrel{n}{=}$$
$$t - B_4,\; t - B_3,\; t - B_2,\; t - B_1,\; t + B_1,\; t + B_2,\; t + B_3,\; t + B_4.$$

Note.—The symbol $\stackrel{n}{=}$ denotes equality of sums of nth powers.

Example.—$7, 24, 25, 34 \stackrel{n}{=} 14, 15, 31, 32$ for $n = 2, 4, 6$ means thus the diophantine system

$$7^2 + 24^2 + 25^2 + 34^2 = 14^2 + 15^2 + 31^2 + 32^2$$
$$7^4 + 24^4 + 25^4 + 34^4 = 14^4 + 15^4 + 31^4 + 32^4$$
$$7^6 + 24^6 + 25^6 + 34^6 = 14^6 + 15^6 + 31^6 + 32^6.$$

The number n in $\stackrel{n}{=}$ indicates that the equality is true for all powers from 1 up to n.

Example.—$1, 5, 9, 17, 18 \stackrel{4}{=} 2, 3, 11, 15, 19$ denotes thus the system

$$1 + 5 + 9 + 17 + 18 = 2 + 3 + 11 + 15 + 19$$
$$1^2 + 5^2 + 9^2 + 17^2 + 18^2 = 2^2 + 3^2 + 11^2 + 15^2 + 19^2$$
$$1^3 + 5^3 + 9^3 + 17^3 + 18^3 = 2^3 + 3^3 + 11^3 + 15^3 + 19^3$$
$$1^4 + 5^4 + 9^4 + 17^4 + 18^4 = 2^4 + 3^4 + 11^4 + 15^4 + 19^4.$$

(2) Consider the relation $7G^2 = H^2 + HJ + J^2$, then we have also $7G+J, -G+2H+3J, -G+3J, -G-2H+J \stackrel{n}{=} G+2H+3J, G+3J, 7G-J, G-2H+J$ for $n = 2, 4, 6$.

Example.—$G = 7$, $H = 1$, $J = 18$ gives the result $67, 49, 47, 9 \stackrel{n}{=} 63, 61, 31, 23$ for $n = 2, 4, 6$.

(3) Given the relation $7G^2 = H^2 + HJ + J^2$, we have also the identity $7G+H+J, -G+3H+3J, -G+3J+H, -G+3H+J \stackrel{n}{=} G+3H+3J, G+3J+H, 7G-H-J, G+3H+J$ for $n = 2, 4, 6$.

Example.—$G = 7$, $H = 1$, $J = 18$ gives, after reduction of the bases by 2, the solution $34, 25, 24, 7 \stackrel{n}{=} 32, 31, 15, 14$ for $n = 2, 4, 6$.

2. (1) Gloden mentions the theorem that,

if $a_1, a_2, a_3 \stackrel{p}{=} b_1, b_2, b_3$ for $p = 2, 4$ with the condition $a_1 + a_2 - a_3 = 2(b_1 + b_2 - b_3)$, then it is also true that $5a_1 - 3a_2 + 3a_3$, $7a_1 - a_2 + a_3$, $-3a_1 + 5a_2 + 3a_3$, $-a_1 + 7a_2 + a_3$, $2(b_1 + b_2 + 3b_3) \stackrel{n}{=} a_1 + a_2 + 7a_3$, $3a_1 + 3a_2 + 5a_3$, $2(3b_1 - b_2 + b_3)$, $2(-b_1 + 3b_2 + b_3)$, $3(a_1 + a_2 - a_3)$ for $n = 1, 3, 5, 7$, whereby a parametric solution of $\sum_{i=1}^{5} C_i^n = \sum_{i=1}^{5} D_i^n$ is obtained for $n = 1, 3, 5, 7$.

Example.—From $6, 23, 25 \stackrel{p}{=} 10, 19, 27$ for $p = 2, 4$ follows, after reducing the calculated bases by 4, the result

$$9, 11, 43, 45, 55 \stackrel{n}{=} 51, 53, 19, 37, 3$$

for $n = 1, 3, 5, 7$.

(2) Further solutions of $\sum_{i=1}^{5} U_i^n = \sum_{i=1}^{5} V_i^n$ for $n = 1, 3, 5, 7$ are given by the formulæ

(α) $-21x^2 + 38xy - 22y^2$, $-7x^2 - 38xy + 50y^2$, $7x^2 - 62xy + 30y^2$, $9x^2 - 14xy - 50y^2$, $21x^2 - 36xy - 30y^2 \stackrel{n}{=} -19x^2 + 36xy - 62y^2$, $-5x^2 - 42xy + 62y^2$, $5x^2 - 8xy - 22y^2$, $9x^2 - 66xy + 42y^2$, $19x^2 - 32xy - 42y^2$ for $n = 1, 3, 5, 7$;

(β) $6x^2 + 18xy - 15y^2$, $10x^2 - 20xy + 15y^2$, $26x^2 - 10xy - 9y^2$, $30x^2 - 42xy + 9y^2$, $34x^2 - 30xy - y^2 \stackrel{n}{=} 6x^2 - 4xy - y^2$, $10x^2 + 8xy - 11y^2$, $26x^2 - 32xy + 5y^2$, $30x^2 - 20xy - 5y^2$, $34x^2 - 36xy + 11y^2$ for $n = 1, 3, 5, 7$.

Example.—In (β), put $x = 3$, $y = 1$, we obtain $41, 103, 143, 205, 209 \stackrel{n}{=} 45, 93, 153, 195, 215$ for $n = 1, 3, 5, 7$.

Note.—From (β), follows the relation $2a^2 - 5ab + 2b^2$, $4a^2 + 12ab - 10b^2$, $6a^2 - 7ab + 5b^2$, $12a^2 - 19ab + 5b^2$, $14a^2 - 2ab - 7b^2$, $16a^2 - 21ab + 8b^2$, $18a^2 - 12ab - 3b^2 \stackrel{k}{=} 2a^2 + 6ab - 5b^2$, $4a^2 - 13ab + 10b^2$, $6a^2 + 7ab - 8b^2$, $12a^2 - 8ab - 2b^2$, $14a^2 - 24ab + 7b^2$, $16a^2 - 7ab - 5b^2$, $18a^2 - 15ab + 3b^2$ for $k = 1, 2, 4, 6, 8$.

Example.—$a=3$, $b=1$ gives

$5, 62, 38, 56, 113, 89, 123 \overset{k}{=} 31, 7, 67, 82, 61, 118, 120$ for $k=1, 2, 4, 6, 8$.

3. (1) If the relation $x, y, z \overset{m}{=} u, v, w$, exists for $m=2, 4$, then Birck gives the relation $x+y+z,\ x+y-z,\ x-y+z,\ -x+y+z,\ 2u,\ 2v,\ 2w \overset{n}{=} u+v+w,\ u+v-w,\ u-v+w,\ -u+v+w,\ 2x,\ 2y,\ 2z$ for $n=1, 2, 4, 6, 8$.

Example.—From $6, 23, 25 \overset{m}{=} 10, 19, 27$ for $m=2, 4$, follows after arranging the bases according to their magnitudes, the solution

$2, 4, 10, 19, 21, 27, 27 \overset{n}{=} 1, 6, 9, 18, 23, 25, 28$ for $n=1, 2, 4, 6, 8$.

Note.—(α) For the equation A, B, C $\overset{m}{=}$ D, E, F for $m=2, 4$, Birck gives the solution $A=9u^3+3u^2-2u-1$, $B=6u^2+2u-1$, $C=9u^3-3u-1$, $D=9u^3+3u^2-4u-1$, $E=6u^2-1$, $F=9u^3-u+1$.

(β) Mahrenholz gives the following solution which is included in (α), with the condition $B-C=D-E$:

$A=2p^2-9pq+15q^2$, $B=6p^2+5pq-9q^2$, $C=10p^2-11pq-3q^2$, $D=2p^2+11pq-15q^2$, $E=6p^2-5pq-9q^2$, $F=10p^2-9pq+3q^2$.

(γ) I give here the following solution included in (α), and satisfying the condition $B-C=E-F$.

$A=3a^2+4ab+5b^2$, $B=2a^2+4ab-10b^2$, $C=a^2-6ab-15b^2$

$D=a^2+4ab+15b^2$, $E=3a^2+6ab-5b^2$, $F=2a^2-4ab-10b^2$.

A four-parametric solution of A, B, C $\overset{m}{=}$ D, E, F for $m=2, 4$ was given by me in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXXI, No. 1, 1939.

(2) For the equation $U_1, U_2, \ldots, U_7 \overset{n}{=} V_1, V_2, \ldots, V_7$ for $n=1, 2, 4, 6, 8$ Mahrenholz gives the following solution:

$U_1=-2x^2+7xy+9y^2$, $U_2=4x^2+22xy-30y^2$, $U_3=6x^2-25xy+21y^2$, $U_4=12x^2-10xy-18y^2$, $U_5=14x^2+3xy-27y^2$, $U_6=18x^2-15xy+3y^2$, $U_7=20x^2-18xy+6y^2$; $V_1=-2x^2+15xy-27y^2$, $V_2=4x^2-18xy+30y^2$, $V_3=6x^2+7xy-3y^2$, $V_4=12x^2+10xy-18y^2$, $V_5=14x^2-25xy+9y^2$ $V_6=18x^2-3xy-21y^2$, $V_7=20x^2-22xy-6y^2$.

A different solution for $U_1, U_2, \ldots, U_7 \overset{n}{=} V_1, V_2, \ldots, V_7$ for $n=1, 2, 4, 6, 8$ is given by the identity $6y^2-3y,\ 4y^2-5y-6,\ -3y+4,\ 2y^2+5y+2,\ -4y^2+6,\ -2y^2+8y+4,\ 6y^2-2y+2 \overset{n}{=} 3y+6,\ -6y^2+5y+4,\ 2y^2-5y+2,\ 4y^2+3y,\ 4y^2-8y-2,\ 6y^2-4,\ 2y^2+2y+6$ for $n=1, 2, 4, 6, 8$.

Example.—$y=4$ gives

$42, 19, -4, 27, -29, 2, 45 \overset{n}{=} 9, -36, 7, 38, 15, 46, 23$

$n=1, 2, 4, 6, 8$.

(3) From my 4-parametric solution to A, B, C $\stackrel{m}{=}$ D, E, F for $m=2, 4$ published in the *Bulletin of the Calcutta Mathematical Society* (*loc. cit.*), I write down the following identity for the solution of $U_1, U_2, \ldots, U_7 \stackrel{n}{=} V_1, V_2, \ldots, V_7$ for $n = 1, 2, 4, 6, 8$.

$$[9z^2(x^2-xy)+3zu(-2xy-y^2)+u^2(-x^2-3xy-2y^2)],$$
$$[3z^2(x^2-3xy+2y^2)+3zu(-2xy+y^2)+3u^2(-x^2-xy)],$$
$$[3z^2(x^2+xy-2y^2)+zu(4x^2-7y^2)+u^2(x^2-xy-2y^2)],$$
$$[3z^2(x^2-xy)+zu(-4x^2+y^2)+u^2(x^2+xy)],$$
$$[6z^2(-x^2+xy)+2zu(2x^2+3xy-2y^2)+2u^2(x^2-y^2)],$$
$$[6z^2(-x^2+y^2)-2zu(2x^2-3xy-2y^2)+2u^2(x^2+xy)],$$
$$[6z^2(x^2-2xy+y^2)+2zu.3y^2+2u^2(x^2+2xy+y^2)] \stackrel{n}{=}$$
$$[3z^2(-x^2-xy+2y^2)+3zu(2xy+y^2)+3u^2(x^2+xy)],$$
$$[9z^2(-x^2+xy)+3zu(2xy-y^2)+u^2(x^2-xy-2y^2)],$$
$$[3z^2(x^2-xy)+zu(4x^2-y^2)+u^2(x^2+xy)],$$
$$[3z^2(x^2-3xy+2y^2)+zu(-4x^2+7y^2)+u^2(x^2+3xy+2y^2)],$$
$$[6z^2(x^2-xy)+2zu(2x^2-3xy-2y^2)-2u^2(x+y)^2],$$
$$[6z^2(x-y)^2-2zu(2x^2+3xy-2y^2)+2u^2(-x^2-xy)],$$
$$[6z^2(x^2-y^2)-2zu.3y^2+2u^2(x^2-y^2)] \text{ for } n=1, 2, 4, 6, 8.$$

Example.—$x=7$, $y=19$, $z=4$, $u=-3$ gives $-3, 529, -289, -243, 436, 482, 0 \stackrel{n}{=} 459, 459, -23, 23, 240, 286, -532$ for $n=1, 2, 4, 6, 8$.

(4) The relation $C_1, C_2, \ldots, C_{14} \stackrel{9}{=} D_1, D_2, \ldots, D_{14}$.

Starting with the relation $x, y, z \stackrel{m}{=} u, v, w$ for $m=2, 4$ and solving after Birck's method (see § 3 (1)), the equation $U_1, U_2, \ldots, U_2 \stackrel{9}{=} V_1, V_2, \ldots, V_7$ for $n=1, 2, 4, 6, 8$, we obtain by the theory of simultaneous identities, the identity $-x-y-z$, $-x-y+z$, $-x+y-z$, $x-y-z$, $-2u$, $-2v$, $-2w$, $2w$, $2v$, $2u$, $y+z-x$, $x+z-y$, $x+y-z$, $x+y+z \stackrel{9}{=} -u-v-w$, $-u-v+w$, $-u+v-w$, $u-v-w$, $-2x$, $-2y$, $-2z$, $2z$, $2y$, $2x$, $-u+v+w$, $u-v+w$, $u+v-w$, $u+v+w$, wherein the condition $x, y, z \stackrel{m}{=} u, v, w$ for $m=2, 4$ must be satisfied.

(5) If in the formula in (4), the condition $y-x=w-v$ is satisfied, then we obtain from the known theory of simultaneous identities, a solution of the equation $E_1, E_2, \ldots, E_{2?} \stackrel{10}{=} F_1, F_2 \ldots, F_{20}$. Thus, given the relation $a_1, a_2, \ldots, a_r \stackrel{k}{=} b_1, b_2, \ldots, b_r$ for the powers from 1 up to k, we get for arbitrary h the relation

$a_1, a_2, \ldots, a_r, b_1+h, b_2+h, \ldots, b_r+h \stackrel{k+1}{=} b_1, b_2, \ldots, b_r, a_1+h, a_2+h, \ldots, a_r+h$. This relation is true for powers from 1 up to $k+1$, it determines therefore equality of sums of all powers from 1 up to $k+1$. We

apply this fact to the formula in (4) and obtain, since the terms $-x+y-z$, $-2v$, $2w$, $y+z-x$, $-u-v+w$, $-2x$, $2y$, $u-v+w$ can be removed from either side, the identity $-x-y-z$, $-x-y+z$, $x-y-z$, $-2u$, $-2w$, $2v$, $2u$, $x+z-y$, $x+y-z$, $x+y+z$, $-u-3v+w$, $-u-3v+3w$, $u-3v+w$, $-4x+2y$, $-2z+2y-2x$, $2z+2y-2x$, $4y-2x$, $-u-v+3w$, $u-3v+3w$, $u-v+3w \stackrel{10}{=} -u-v-w$, $-u+v-w$, $u-v-w$, $-2y$, $-2z$, $2z$, $2x$, $-u+v+w$, $u+v-w$, $u+v+w$, $-3x+y-z$, $-3x+y+z$, $-3x+3y-z$, $-2u+2w-2v$, $-4v+2w$, $4w-2v$, $2u+2w-2v$, $3y+z-3x$, $-x+3y-z$, $-x+3y+z$, wherein the condition $x, y, z \stackrel{n}{=} u, v, w$ for $n=2, 4$, is satisfied.

Example.—$x=15$, $y=19$, $z=31$, $u=9$, $v=25$, $w=29$ gives -65, -3, -35, -18, -58, 50, 18, 27, 3, 65, -55, 3, -37, -22, -54, 70, 46, 53, 21, $71 \stackrel{10}{=} -63$, -13, -45, -38, -62, 62, 30, 45, 5, 63, -57, 5, -19, -10, -42, 66, 26, 43, 11, 73.

Note.—When the exponents which specify the powers for which equality of sums exists in a simultaneous identity, form an arithmetic progression with first term 1 and difference 1, then we can add to the terms of this identity an arbitrary (but always the same) number, without destroying the validity of the simultaneous identity. Consider thus the simultaneous identity $A_1, A_2, \ldots, A_r \stackrel{n}{=} B_1, B_2, \ldots, B_r$; then for arbitrary k, we have also the identity $A_1+k, A_2+k, \ldots, A_r+k \stackrel{n}{=} B_1+k, B_2+k, \ldots, B_r+k$.

We apply this fact to our previous numerical solution and putting $k=66$, we get the result 1, 63, 31, 48, 8, 116, 84, 93, 69, 131, 11, 69, 29, 44, 12, 136, 112, 119, 87, $137 \stackrel{10}{=} 3$, 53, 21, 28, 4, 128, 96, 111, 71, 129, 9, 71, 47, 56, 24, 132, 92, 109, 77, 139.

(6) Solution of $G_1, G_2, \ldots, G_{10} \stackrel{n}{=} H_1, H_2, \ldots, H_{10}$ for $n=1, 3, 5, 7, 9$.

(α) Given $p_1, p_2, \ldots, p_r \stackrel{d}{=} q_1, q_2, \ldots, q_r$ for the first d powers from 1 up to d, then we have for arbitrary s, $p_1-s, p_2-s, \ldots, p_r-s \stackrel{d}{=} q_1-s, q_2-s, \ldots, q_r-s$. We apply this theorem to the solution formula for $E_1, E_2, \ldots, E_{20} \stackrel{10}{=} F_1, F_2, \ldots, F_{20}$ in (5) and obtain by putting $s=-x+y=w-v$, the formula

(β) $-2y-z$, $-2y+z$, $2x-2y-z$, $-2u-w+v$, $-3w+v$, $3v-w$, $2u-w+v$, $2x+z-2y$, $2x-z$, $2x+z$, $-u-2v$, $-u-2v+2w$, $u-2v$, $-3x+y$, $-2z+y-x$, $2z+y-x$, $3y-x$, $-u+2w$, $u-2v+2w$, $u+2w \stackrel{10}{=} -u-2w$, $-u+2v-2w$, $u-2w$, $x-3y$, $-2z+x-y$, $2z+x-y$, $3x-y$, $-u+2v$, $u+2v-2w$, $u+2v$, $-2x-z$, $-2x+z$, $-2x+2y-z$, $-2u+w-v$, $-3v+w$, $3w-v$, $2u+w-v$, $2y+z-2x$, $2y-z$, $2y+z$.

From this follows

(γ) $2y+z,\ 2y-z,\ 2y+z-2x,\ 2u+w-v,\ 3w-v,\ u+2v,\ u+2v-2w,$ $-u+2v,\ 3x-y,\ 2z+x-y \stackrel{n}{=} u+2w,\ u-2v+2w,\ -u+2w,\ 3y-x,$ $2z+y-x,\ 2x+z,\ 2x-z,\ 2x+z-2y,\ 2u-w+v,\ 3v-w$ for $n=1, 3, 5, 7, 9$.

Example.—$x=15$, $y=19$, $z=31$, $u=9$, $v=25$, $w=29$ gives 69, 7, 39, 22, 62, 59, 1, 41, 26, 58 $\stackrel{n}{=}$ 67, 17, 49, 42, 66, 61, -1, 23, 14, 46 for $n=1, 3, 5, 7, 9$.

4. (1) If the relation $C_1, C_2, \ldots, C_r \stackrel{n}{=} D_1, D_2, \ldots, D_r$ for $n=1, 3, \ldots, 2p+1$ holds, in other words, if in the simultaneous identity $C_1, C_2, \ldots, C_r \stackrel{n}{=} D_1, D_2, \ldots, D_r$, the n exponents form an arithmetic series with first term 1 and difference 2, then for arbitrary T, we have the identity $T-C_1, T-C_2 \ldots, T-C_r, T+D_1, T+D_2, \ldots, T+D_r \stackrel{k}{=} T-D_1, T-D_2, \ldots, T-D_r, T+C_1, \ldots, T+C_r$ for $k=1, 2, \ldots, 2p+1, 2p+2$.

We apply this theorem to the formula $\sum_{i=1}^{5} U_i^n = \sum_{i=1}^{5} V_i^n$ for $n=1, 3, 5, 7$. We obtain in this way from the formula in §2 (2), (α), the following identity (2) $0,\ 5x^2+17xy-42y^2,\ 7x^2-12xy-10y^2,\ 12x^2+5xy-52y^2,\ 19x^2-34xy+10y^2,\ -x^2+3xy-32y^2,\ 6x^2-35xy+4y^2,\ 13x^2-47xy-6y^2,$ $14x^2-23xy-46y^2,\ 20x^2-34xy-36y^2 \stackrel{8}{=} 20x^2-35xy-10y^2,\ 13x^2+3xy-46y^2,\ 6x^2+15xy-36y^2,\ 5x^2-9xy+4y^2,\ -x^2+2xy-6y^2,\ 2xy-52y^2,$ $7x^2-37xy-10y^2,\ 12x^2-20xy-32y^2,\ 14x^2-49xy,\ 19x^2-32xy-42y^2$.

Example.—$x=3$, $y=1$ gives, when the result is raised so as to have only positive numbers, after arranging these numbers according to magnitude, the numerical solution 11, 26, 28, 58, 69, 75, 100, 112, 129, 137 $\stackrel{8}{=}$ 12, 20, 37, 49, 74, 80, 91, 121, 123, 138.

The formula in II (2), (β) gives:

$0,\ 2x^2-8xy+8y^2,\ 4x^2-2xy+3y^2,\ 12x^2-22xy+11y^2,\ 14x^2-16xy+6y^2,$ $20x^2-9xy-2y^2,\ 22x^2-28xy+13y^2,\ 30x^2-23xy+y^2,\ 32x^2-39xy+10y^2,$ $34x^2-33xy+5y^2 \stackrel{8}{=} 14x^2-27xy+13y^2,\ 12x^2-8xy-2y^2,\ 4x^2-13xy+10y^2,$ $2x^2+3xy+y^2,\ -3xy+6y^2,\ 20x^2-20xy+5y^2,\ 22x^2-14xy,\ 30x^2-34xy+8y^2$ $32x^2-28xy+3y^2,\ 34x^2-36xy+11y^2$.

Example.—$x=3$, $y=1$ gives, when the numbers are arranged according to their magnitudes, the solution 0, 2, 33, 53, 84, 127, 151, 181, 202 212 $\stackrel{8}{=}$ -3, 7, 28, 58, 82, 125, 156, 176, 207, 209.

GEOMETRY AS THE STUDY OF SPACE-STRUCTURE

BY

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1. To the question, "What is geometry?" it is not possible to find a simple answer. Geometry is a rich field of human activity extending from over 2,000 years ago to the present day, and comprising such diverse topics as Euclidean Synthetic geometry, the Non-Euclidean geometries of Lobatchevski and Riemann, Projective geometry, the Analytical geometry of Descartes which may be extended to any field, Finite geometries, Differential geometry and Topology to mention only a few of the major developments. A fairly satisfactory definition of the geometry of his day was given by Felix Klein in his Erlanger Program (1872) when he defined geometry as the study of the properties of a system of elements which are invariant for all the transformations of any group which carries those elements into themselves. While this definition has the merit of covering the well known types of geometry, including Topology, it turns out that the group associated with the differential geometry of Riemann and its later developments which sprang into prominence with the advent of the general theory of relativity, is the one element group consisting of the identity only. The association with group theory, in this case, does not throw any light on the nature of the geometry apart from merely tolerating it!

The modern view of geometry may be expressed by the statement that Geometry is the study of the structure of space. Space is considered as the totality of certain elements, called for convenience "points" without any commitments thereby, with a certain (arbitrary) structure imposed thereon. This structure may be imposed for example, by means of certain undefined terms like "point," "line," "incident," "between," "congruent," "parallelism," etc.,* and certain initial unproved theorems or postulates. It is then the business of geometry to work out the consequences of our postulates, and bring to view the space-structure whose germ is already contained in our postulates. At some stage, it may be possible to associate some kind of number-system (or co-ordinates) with the elements of the geometry, the properties of the number system depending on the initial postulates. Thus it is known that the truth of Pappus's theorem is connected with the commutativity of multiplication of the numbers introduced. On the other hand, we could reverse this process and start with a ground field K (in which multiplication may or may not be commutative) and define

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* Vide Hilbert: *Foundations of Geometry.*

a " point" and the other elements of the geometry by means of sets of numbers in K, and thereby impose a " structure " on the geometry.

Thus if we define an ordered set of n numbers $(x_1, x_2, \ldots, x_n)$ as a "point," we get a Euclidean space of n-dimensions. On the other hand by defining a " point " to be a set of $(n+1)$ numbers of the field $(x_0, x_1, \ldots, x_n)$ *not all simultaneously* zero, and regarding the above set as defining the *same point* as $(x_0\lambda, x_1\lambda, \ldots, x_n \lambda)$ where $\lambda \neq 0$ is any number in K we get a right-handed projective geometry $P_n^r(K)$ of n dimensions over K. If on the other hand we regard $(x_0, x_1, \ldots, x_\mu)$ as defining the same point as $(\mu x_1, \mu x_0, \ldots, \mu x_n)$, $\mu \neq 0$ we get a left-handed projective geometry $P_n^l(K)$. It should be noted that in each case the equivalence of co-ordinates defining the same point is a symmetric, reflexive, transitive relation which separates all $(n+1)$ – ads into equivalent sets.

2. It is noteworthy that even in merely defining a point, we have already imposed a structure on the space. Thus if we wish to change the " co-ordinates " of the points, so that the correspondence is one-one, and confine ourselves say, to the field of complex numbers and to analytic transformations therein, we are confronted with the affine group in the first case and the projective group in the second. Since the maximal automorphic group of the elements fixes the maximal geometry that can be imposed on the space, it would appear that in defining the point by means of a particular number system, we have already chosen our maximal geometry. The significance of Klein's approach is that we treat different systems of allowable co-ordinates as convenient alternative scaffoldings to erect our structure, but take care to see that they do not get built into our structure and are removable without damage to it.*

In the case of a non-commutative field K, it is reasonable to choose the group of transformations of the right projective geometry as $x_r' = \Sigma a_{rs} x_s$ and of the left-geometry as $x_r' = \Sigma x_s b_{sr}$, the matrices $[a_{rs}]$ and $[b_{rs}]$ being non-singular.

A further generalisation leads us to the " product spaces". Thus take the product of a projective m-space $(x_0, x_1, \ldots, x_m)$ and a projective n-space $(y_0, y_1, \ldots, y_n)$ over a commutative field K. It is a space whose points have co-ordinates $(x_0, x_1, \ldots, x_m;\ y_0, y_1, \ldots, y_n)$, this point being considered

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* An axiomatic formulation of the ideas of the Erlanger Program, will be found in Weyl : *The Classical Groups*, p. 17.

The attempt at fixing a unique set of cordinates for a point in the projective plane, by imposing an additional condition like $x + y + z = 1$ for areal cordinates so common in text-books. is misleading and deserves condemnation It serves no purpose and leads to absurdities when dealing with points at infinity which are respectable citizens of the projective plane.

the same as $(\lambda x_0, \lambda x_1, \ldots, \lambda x_m; \mu y_0, \mu y_1, \ldots, \mu y_n)$, $\lambda, \mu \neq 0$. The transformation group of this space is the one induced by independent projective transformations in the respective spaces. One of the simplest of these is the double binary space corresponding to $m=2$, $n=2$.

In the case of the spaces of differential geometry the structure is imposed on the space through the metrical tensor g_{ij} or through the affine-connection, the imposition of the structure being a direct act in these cases.

3. Whatever the structure imposed, whether through a co-ordinate system and an automorphic group, or by its local properties, an important question is to determine the topological properties of the space as a whole, such as its connectivity, its one-sidedness or otherwise and so on.

By way of illustration, let us take a projective 2-dimensional space over the field of real numbers. With the ratios $\alpha:\beta:\gamma$ of three numbers not all zero, we may associate a line $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$ through the origin in a $(1-1)$ continuous correspondence.

By taking the section of these lines by a sphere, centre O, we have a representation on the points of a sphere, diametrically opposite points being considered identical, (since they are the intersections with the same line and correspond to one set $\alpha:\beta:\gamma$. We may cut out one hemisphere so as to avoid pairs of diametrically opposite points on the surface, but such pairs will still persist on the boundary. Since we may by continuous deformation convert a hemispherical surface into a plane circle, we may say that

The real projective plane is topologically equivalent to the interior and boundary of a plane circle, diametrically opposite points on the boundary being considered identical. (3.1)

By a " topological image," we mean that there is a bi uniform, *i.e.*, a one-one correspondence between the points of the projective plane and its image which is continuous both ways (bi-uniform and bi-continuous) The actual identification of the diametrically opposite points cannot be brought about in 3-space, so that there is no simple model here of the whole projective plane. However, we can get a model of a *part* of the projective plane by taking the part of the circle bounded by two parallel lines equidistant from the centre. By a continuous deformation, we get a rectangle $ABCD$, in which A and C correspond to diametrically opposite points as also B and D. By giving a half-twist to the strip and pasting BC over AD so that B falls on D and C on A, we get the Möbius sheet,—as such a surface is called—as a topological image of a part of the projective plane. It is easy to verify from this model (1) that there is only one edge for the sheet, (ii) that the surface is one-sided, that is, it is possible to pass continuously from one side to the other without crossing the edge, (iii) that a closed curve does not

necessarily divide the plane into two regions with the result that the edge has to be crossed in passing from one to the other. This last property can be seen, for instance, by cutting the strip all along the midway line, when it will be seen that the sheet still remains connected.

As contrasted with (3.1)

the real Euclidean plane is topologically equivalent to the open set of points (x, y) *for which* $x^2 + y^2 < 1$. (3.2)

This is easily deducible from the fact that the region $0 \leqslant r < \infty$ can be mapped uniformly and continuously on $0 \leqslant r' < 1$ and *vice versa.*

Lastly let us take the double binary space in which a "point" is defined to be the set of numbers $(x_1, x_2;\ y_1, y_2)$, this point being the same as $(\lambda x_1, \lambda x_2;\ \mu y_1, \mu y_2)$, $\lambda \neq 0$, $\mu \neq 0$. Now the single binary domain—the projective 1-space—can be mapped topologically on a circle by means of the correspondence, $x_1 = \lambda \cos \theta$, $x_2 = \lambda \sin \theta$, $0 \leqslant \theta \leqslant 2\pi$, the points corresponding to $\theta = 0$ and $\theta = 2\pi$ being the same. Hence a topological picture of a double binary space is given by the surface of an anchor ring, on which every point is fixed by two angles θ, ϕ between 0 and 2π, the points corresponding to 0 and 2π being the same. A projective transformation of any one of the binary variables, corresponds to an interchange of the circles of either system among themselves.

We may cut out the anchor ring, straighten it out, and thus get a cylindrical image provided the two cut ends are identified. We thus get another picture of the double binary space on a hyperboloid of one sheet in a projective 3-space, the identification of the cut ends being effected automatically owing to there being only one point at infinity on every generator. That the hyperboloid $xy = zw$ is a true topological picture, may be seen by introducing homogeneous co-ordinates $x_1 : x_2$ and $y_1 : y_2$ for the two systems of generators by the formulæ

$$x = \lambda x_1 y_1, \quad y = \lambda x_2 y_2, \quad z = \lambda x_1 y_2, \quad w = \lambda x_2 y_1.$$

Each system of generators constitutes a projective space, and any two "points" one from each of the two one-dimensional spaces (that is to say, two generators one from each system), defines a "point" of the product manifold, namely the hyperboloid.

A simple example of a binary-ternary product manifold is given by the line-elements on a projective plane, for it is the topological product of the "points" of the projective plane, and the binary domain of θ, $0 \leqslant \theta \leqslant 2\pi$, 0 and 2π being identified. In the same manner it is easy to see that the line elements of a quadric is an example of a triple binary domain.

NOTES AND DISCUSSIONS

On Kemmer's Identity in Combinatory Functions

Introduction.—Using some elementary properties of Γ-functions and the hypergeometric function ${}_2F_1$, Kemmer* has proved that

$$\binom{2s}{s \pm k} = \sum_{0 \leqslant p \leqslant \frac{1}{2}(s-k)} 2^{s-2p-k} \binom{s}{p}\binom{s-p}{p+k}. \tag{1}$$

It will be readily seen that, in this form, the identity holds only for non-negative integral values of s and k, $s \geqslant k$.

It was thought worth while to establish the identity by elementary methods. A number of such proofs, and some generalizations have been worked out by some of my friends, to whom I communicated the problem. They are collected together, below.

C. N. S.

I

By Hansraj Gupta, Hoshiarpur

I shall first give an *ab initio* proof of the slightly more general identity†

$$\binom{2s}{r} = \sum_{p \geqslant 0} 2^{r-2p} \binom{s}{r-p}\binom{r-p}{p}, \tag{2}$$

which holds for non-negative integral values of r and *all s*. The method is used to obtain a similar identical relation for $\binom{ns}{r}$, n being an integer $\geqslant 2$.

Kemmer states that he arrived at his identity by trial and error and first proved it by induction. He does not seem to have published, however, his inductive proof. I, therefore, give here an inductive proof of (2) as well.

2. Proof of (2).

First consider s to be a positive integer.

Divide the $2s$ articles into s pairs in any way.

To select r articles from the s pairs, we first choose $(r-p)$ pairs. This can be done in $\binom{s}{r-p}$ ways. From the $(r-p)$ pairs we next choose p

* N. Kemmer, "The Algebra of Meson Matrices," *Proc. Cambr. Phil. Soc.*, 1943, **39**, 189–96.

† Writing $r = s - k$, $\binom{s}{r-p}\binom{r-p}{p} = \frac{s!}{(s-p-k)!\,(p+k)!} \cdot \frac{(s-p-k)!}{p!(s-k-2p)!}$

$$= \frac{s!}{p!\,(s-p)!} \cdot \frac{(s-p)!}{(p+k)!\,(s-2p-k)!}$$

$$= \binom{s}{p}\binom{s-p}{p+k}.$$

pairs. This can be done in $\binom{r-p}{p}$ ways. Taking the $2p$ articles in these p pairs, we are left to select $(r-2p)$ articles, one from each of the remaining $(r-2p)$ pairs. This can obviously be done in 2^{r-2p} ways. Hence, the result.

Since $\binom{t}{r}$ is an algebraic expression of the rth degree in t, (2) holds for all values of s and for non-negative integral values of r.

3. To find an expression for $\binom{ns}{r}$, we place the ns articles in s heaps of n articles each. If $g_1, g_2, g_3, \cdots, g_n$ be any n non-negative integers satisfying the two relations:

$$g_1+g_2+g_3+\cdots+g_n=r-p,\ p\geqslant 0;$$

and

$$g_1+2g_2+3g_3+\cdots+ng_n=r; \tag{E}$$

then to select r articles from the s heaps, we first choose $(r-p)$ heaps. This can be done in $\binom{s}{r-p}$ ways.

Place g_1 of these $(r-p)$ heaps in group 1, g_2 in group 2, g_3 in group 3, and so on. This can be done in

$$\frac{(r-p)\,!}{g_1!\ g_2!\ g_3!\cdots g_n!}$$

ways. Now from each of the g_k heaps in group k, select k articles. This can be done in

$$\left\{\binom{n}{k}\right\}^{g_k}$$

ways. Hence, we get

$$\binom{ns}{r}=\sum_{p\geqslant 0}\left\{(r-p)\,!\prod_{k=1}^{n}\frac{\binom{n}{k}^{g_k}}{g_k!}\right\}\binom{s}{r-p}, \tag{3}$$

where the g's run through all the non-negative solutions in integers of the equations (E).

4. The following is an inductive proof of (2).

Suppose that when $0\leqslant r\leqslant j$,

$$\binom{2s}{r}=\sum_{p\geqslant 0}2^{2r-p}\binom{r-p}{p}\binom{s}{r-v},$$

then

$$\binom{2s}{j+1}=\frac{2s-j}{j+1}\sum_{p\geqslant 0}2^{j-2p}\binom{j-p}{p}\binom{s}{j-p}$$

Now

$$\frac{2s-j}{j+1}=\frac{2\,(s-j+p)+(j-2p)}{j+1}.$$

Therefore,

$$\binom{2s}{j+1} = \sum_{p \geqslant 0} \left\{ \frac{j-2p+2}{j+1} . 2^{j-2p+2} \binom{j-p+1}{p-1} \binom{s}{j-p+1} \right.$$

$$\left. + \frac{2(s-j+p)}{j+1} . 2^{j-2p} \binom{j-p}{p} \binom{s}{j-p} \right\}, \text{ where } \binom{j+1}{-1} = 0;$$

$$= \sum_{p \geqslant 0} \left\{ 2^{j-2p+2} . \frac{p}{j+1} . \binom{j-p+1}{p} \binom{s}{j-p+1} \right.$$

$$\left. + 2^{j-2p+1} . \frac{j-2p+1}{j+1} . \binom{j-p+1}{p} \binom{s}{j-p+1} \right\},$$

$$= \sum_{p \geqslant 0} 2^{j-2p+1} \binom{j-p+1}{p} \binom{s}{j-p+1};$$

and the result follows by induction.

5. The problem could also be considered in the following manner. Suppose

$$\binom{ns}{r} \equiv a_r \binom{s}{r} + a_{r-1} \binom{s}{r-1} + \cdots + a_2 \binom{s}{2} + a_1 \binom{s}{1}, \quad r > 0;$$

where the a's are certain constants depending on n and r only. Then, putting $s = 1, 2, 3, \cdots, r$ in turn, it can be shown that

$$a_k = \binom{kn}{r} - \binom{k}{1} \binom{(k-1)n}{r} + \binom{k}{2} \binom{(k-2)n}{r} - \cdots + (-1)^{k-1} \binom{k}{k-1} \binom{n}{r}$$

Several results may be obtained by comparing the coefficients in the two expressions for $\binom{ns}{r}$.

II

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PART I

1. NOTATIONS.

$\binom{m}{p} = \frac{m(m-1)(m-2)\ldots.p \text{ factors}}{p!}$, where m is any number and p is a positive integer.

$$\binom{m}{p} = 1 \text{ if } p = 0, \binom{m}{p} = 0 \text{ if } p \text{ is negative.}$$

If $|m-p| \equiv 0 \pmod{b}$, then $\left\{ {m \atop p} \right\}_b = m(m-b)(m-2b)..p;$

$$\left\{ {m \atop p} \right\}_b = 1 \text{ if } m < p.$$

2. A THEOREM FROM FINITE DIFFERENCES.

The first element of the nth *order of differences obtained from the series* $u_0, u_1, u_2, u_3, \ldots,$ is $\sum\limits_{0 \leqslant p \leqslant n} (-)^p\, u_p \binom{n}{p}$. (1)

3. TWO AUXILIARY THEOREMS.

$$\sum_{0 \leqslant p \leqslant k} \left\{ \begin{matrix} 2k-1 \\ 2k-2p+1 \end{matrix} \right\}_2 \left\{ \begin{matrix} 2m-2k \\ 2m-4k+2p+2 \end{matrix} \right\}_2 \binom{k}{p} = \left\{ \begin{matrix} 2m-1 \\ 2m-2k+1 \end{matrix} \right\}_2 \tag{2}$$

and

$$\sum_{0 \leqslant p \leqslant k} \left\{ \begin{matrix} 2k+1 \\ 2k-2p+3 \end{matrix} \right\}_2 \left\{ \begin{matrix} 2m-2k-2 \\ 2m-4k+2p \end{matrix} \right\}_2 \binom{k}{p} = \left\{ \begin{matrix} 2m-1 \\ 2m-2k+1 \end{matrix} \right\}_2. \tag{3}$$

Each of these theorems follows readily from (1), u_p being

$$(-)^p \left\{ \begin{matrix} 2k-1 \\ 2k-2p+1 \end{matrix} \right\}_2 \left\{ \begin{matrix} 2m-2k \\ 2m-4k+2p+2 \end{matrix} \right\}_2 \text{ and}$$

$$(-)^p \left\{ \begin{matrix} 2k+1 \\ 2k-2p+3 \end{matrix} \right\}_2 \left\{ \begin{matrix} 2m-2k-2 \\ 2m-4k+2p \end{matrix} \right\}_2 \text{ respectively.}$$

Note.—These theorems may also be proved in another way. Each is an equation of the kth degree in m and by difference formulæ, it can be shown that it is satisfied for $k+1$ values of m, *viz*, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2k-1}{2}$, and 0. Hence it is an identity.

4. KEMMER'S THEOREM.

$$\binom{2m}{m \pm k} = \sum_{0 \leqslant p \leqslant \frac{1}{2}(m-k)} 2^{m-2p-k} \binom{m}{p} \binom{m-p}{p+k}. \tag{4}$$

Proof.—If $\frac{1}{2}(m-k) = n$, the theorem may be written

$$\binom{2m}{2n} = \sum_{0 \leqslant p \leqslant n} 2^{2n-2p} \binom{m}{p} \binom{m-p}{2n-2p}, \tag{5}$$

where $2n$ is an even or odd integer.

Case (i), *when* $2n$ *is an even integer.*—Let us consider the particular case when $2n = 6$. Then the right side of (5) is

$$\frac{2m\,(2m-2)\,(2m-4)}{6\,!} \sum_{0 \leqslant p \leqslant 3} \left\{ \begin{matrix} 5 \\ 7-2p \end{matrix} \right\}_2 \left\{ \begin{matrix} 2m-6 \\ 2m+2p-10 \end{matrix} \right\}_2 \binom{3}{p}$$

$$= \binom{2m}{6}, \text{ by (2).}$$

So the theorem holds in this particular case. The general case may be similarly treated.

Case (ii), *when* $2n$ *is an odd integer.*—Suppose $2n=5$, then the right side of (5) is $\sum_{0 \leqslant p \leqslant 2} 2^{5-2p} \binom{m}{p} \binom{m-p}{5-2p}$, the terms when $p=3, 4$, etc., all vanish.

$$= \frac{2m(2m-2)(2m-4)}{5!} \sum_{0 \leqslant p \leqslant 2} \left\{ \begin{matrix} 5 \\ 7-2p \end{matrix} \right\}_2 \left\{ \begin{matrix} 2m-6 \\ 2m+2p-8 \end{matrix} \right\}_2 \binom{2}{p}$$

$$= \binom{2m}{5}, \text{ by (3).}$$

So the theorem is true in this particular case. The general case may be similarly proved.

PART II

A Theorem on Higher Differences

In this part is given a theorem of which the identity given by Kemmer is a parti ular case. In proving this theorem, two other theorems have been obtained.

5. NOTATIONS.—If m and c be any numbers and $|m-c| \equiv 0 \pmod{b}$, then according as $m >= $ or $< c$

$$\begin{bmatrix} m \\ c \end{bmatrix}_b = (a^m-1)(a^{m-b}-1)(a^{m-2b}-1)\cdots(a^c-1),\ a^m-1 \text{ or } 1$$

and
$$\begin{vmatrix} m \\ c \end{vmatrix}_b = (a^m+1)(a^{m-b}+1)(a^{m-2b}+1)\cdots(a^c+1),\ a^m+1 \text{ or } 1.$$

If m and c be positive integers, we write $\begin{bmatrix} m \\ c \end{bmatrix}$ for $\begin{bmatrix} m \\ c \end{bmatrix}_1$ and $\begin{vmatrix} m \\ c \end{vmatrix}$ for $\begin{vmatrix} m \\ c \end{vmatrix}_1$.

We write $${}^mS_p = \frac{(a^m-1)(a^{m-1}-1)(a^{m-2}-1)\ldots p \text{ factors}}{(a^p-1)(a^{p-1}-1)(a^{p-2}-1)\ldots(a-1)} a^{\frac{1}{2}p(p-1)},$$

where m is any number and p is a positive integer. ${}^mS_0=1$, ${}^mS_p=0$ if $p>m$ or <0 and ${}^mS_p=1$ if $m=p=0$. We also denote

$${}^mR_p = \frac{(e^m-1)(e^{m-1}-1)(e^{m-2}-1)\ldots p \text{ factors}}{(e^p-1)(e^{p-1}-1)(e^{p-2}-1)\ldots(e-1)} e^{\frac{1}{2}p(p-1)},$$

where m is any number and $e=1/a^2$;

$${}^mR_0=1,\ {}^mR_p=0 \text{ if } p>m \text{ or } <0 \text{ and } {}^mR_p=1 \text{ if } m=p=0.$$

6. OLD THEOREMS*.

(i) If m is a positive integer, it is evident that

$${}^mS_p = \frac{\begin{bmatrix} m \\ 1 \end{bmatrix}}{\begin{bmatrix} p \\ 1 \end{bmatrix}\begin{bmatrix} m-p \\ 1 \end{bmatrix}} {}^pS_p \tag{1}$$

* Chakrabarti, S., "On the Evaluation of Some Factorable Continuants," *Bull. Cal. Math. Soc.*, 1922–23, **13**, 71–81 and 1923–24, **14**, 91–106.

and it is also evident that ${}^{m}S_p$ is the sum of the products of m factors $1, a, a^2, a^3, \cdots, a^{m-1}$ taken p at a time.

(ii) If $1, a, a^2, a^3, \cdots$ be the successive multipliers, the first element of the nth order of differences obtained from the series $u_0, u_1, u_2, u_3, \cdots$ is

$$\sum_{0 \leqslant p \leqslant n} (-)^p u_p {}^{n}S_p \tag{2}$$

7. THEOREMS.

(i) $$\sum_{0 \leqslant p \leqslant k} \begin{bmatrix} 2k-1 \\ 2k-2p+1 \end{bmatrix}_2 \begin{bmatrix} 2m-2k \\ 2m-4k+2p+2 \end{bmatrix}_2 a^{\lambda}\, {}^{k}R_p = \begin{bmatrix} 2m-1 \\ 2m-2k+1 \end{bmatrix}_2 \tag{3}$$

where $\lambda = k(2k-1) + p^2 - 2pk$

and (ii) $$\sum_{0 \leqslant p \leqslant k} \begin{bmatrix} 2k+1 \\ 2k-2p+3 \end{bmatrix}_2 \begin{bmatrix} 2m-2k-2 \\ 2m-4k+2p \end{bmatrix}_2 a^{\mu}\, {}^{k}R_p = \begin{bmatrix} 2m-1 \\ 2m-2k+1 \end{bmatrix}_2 \tag{4}$$

where $\mu = k(2k+1) + p^2 - 2pk - 2p$.

These theorems follow readily from (2), u_p being

$$(-)^p \begin{bmatrix} 2k-1 \\ 2k-2p+1 \end{bmatrix}_2 \begin{bmatrix} 2m-2k \\ 2m-4k+2p+2 \end{bmatrix}_2 a^{\lambda} \text{ and}$$

$$(-)^p \begin{bmatrix} 2k+1 \\ 2k-2p+3 \end{bmatrix}_2 \begin{bmatrix} 2m-2k-2 \\ 2m-4k+2p \end{bmatrix}_2 a^{\mu} \text{ respectively,}$$

and the successive multipliers are $1, e, e^2, e^3, \cdots$

Corollaries.—If $a = 1$, the theorems (3) and (4) reduce to the auxiliary theorems of §3, part 1.

For ${}^{k}R_p = \binom{k}{p}$ and $\frac{a^l - 1}{a^r - 1} = \frac{l}{r}$ in the limit when $a = 1$.

8. THEOREM.

$${}^{2m}S_{2n} = \sum_{0 \leqslant p \leqslant n} \frac{\left|\begin{matrix} m \\ m-2n+p+1 \end{matrix}\right|}{\left|\begin{matrix} p \\ 1 \end{matrix}\right|} {}^{m}S_p\, {}^{m-p}S_{2n-2p}\, a^l \tag{5}$$

where $l = n(2n-1) - \frac{1}{2}p(p-1)$ and $2n$ is an even or odd integer.

Proof :

Case (i), *when $2n$ is an even integer.*—Let $2n = 6$, then the right side of (5) is

$$\sum_{0 \leqslant p \leqslant 3} \frac{\left|\begin{matrix} m \\ m+p-5 \end{matrix}\right|}{\left|\begin{matrix} p \\ 1 \end{matrix}\right|} {}^{m}S_p\, {}^{m-p}S_{6-2p}\, a^{15-\frac{1}{2}p(p-1)},$$ the terms when $p = 4$, 5, etc., all vanish,

$$= \frac{\begin{bmatrix} 2m \\ 2m-4 \end{bmatrix}_2}{\begin{bmatrix} 6 \\ 1 \end{bmatrix}} a^{15} \left\{ \begin{bmatrix} 2m-6 \\ 2m-10 \end{bmatrix}_2 a^{15} + \ldots + (a^5-1)(a^3-1)(a-1) \right\}$$

$$= \frac{\begin{bmatrix} 2m \\ 2m-4 \end{bmatrix}_2}{\begin{bmatrix} 6 \\ 1 \end{bmatrix}} a^{15} \sum_{0 \leqslant p \leqslant 3} \begin{bmatrix} 5 \\ 7-2p \end{bmatrix}_2 \begin{bmatrix} 2m-6 \\ 2m+2p-10 \end{bmatrix}_2 a^{15+p^2-6p} \, {}^3R_p$$

$$= {}^{2m}S_6, \text{ by (3).}$$

So the theorem is proved in this particular case. The general case may be similarly treated.

Case (ii), *when* $2n$ *is an odd integer.*—If $2n = 5$, the right side of (5) is

$$\sum_{0 \leqslant p \leqslant 2} \frac{\begin{vmatrix} m \\ m+p-4 \end{vmatrix}}{\begin{vmatrix} p \\ 1 \end{vmatrix}} {}^mS_p \, {}^{m-p}S_{5-2p} \, a^{10-\frac{1}{2}p(p-1)},$$ the terms when $p = 3$, 4, etc., all vanish.

$$= \frac{\begin{bmatrix} 2m \\ 2m-4 \end{bmatrix}_2}{\begin{bmatrix} 5 \\ 1 \end{bmatrix}} a^{10} \sum_{0 \leqslant p \leqslant 2} \begin{bmatrix} 5 \\ 7-2p \end{bmatrix}_2 \begin{bmatrix} 2m-6 \\ 2m+2p-8 \end{bmatrix}_2 a^{10+p^2-6p} \, {}^2R_p$$

$$= {}^{2m}S_5, \text{ by (4).}$$

The general case may be similarly proved.

Corollary.—If $a = 1$, theorem (5) reduces to

$$\binom{2m}{2n} = \sum_{0 \leqslant p \leqslant n} 2^{2n-2p} \binom{m}{p} \binom{m-p}{2n-2p}, \qquad (6)$$

which is Kemmer's identity.

III

By K. Venkatachaliengar and V. R. Tiruvenkatachar (Bangalore)

The coefficient of x^{s+k} in the Binomial expansion of $\left(1+\frac{x}{2}\right)^{2s}$ is

$$\binom{2s}{s+k} \frac{1}{2^{s+k}}. \qquad (1)$$

Also,
$$\left(1+\frac{x}{2}\right)^{2s} = \left(1+x+\frac{x^2}{4}\right)^s$$

$$= \sum_{p=0}^{s} \binom{s}{p} \left(1+\frac{x}{4}\right)^{s-p} x^{s-p}. \qquad (2)$$

The coefficient of x^{s+k} in the right side of (2) is

$$\sum_{p=0}^{s} \binom{s}{p} \binom{s-p}{p+k} \cdot 4^{-(p+k)}$$

But it is easily seen that x^{s+k} occurs on the right side of (2) only for those terms for which $s-p \geqslant p+k$, or $2p \leqslant s-k$. Hence, the coefficient of

$$x^{s+k} \text{ is } \sum_{p=0}^{\frac{1}{2}(s-k)} \binom{s}{p}\binom{s-p}{p+k} 2^{(2p+2k)}. \tag{3}$$

Comparing (1) and (3), we have

$$\binom{2s}{s+k} = \sum_{p=0}^{\frac{1}{2}(s-k)} \binom{s}{p}\binom{s-p}{p+k} 2^{2s-sp-k}, \tag{4}$$

which is Kemmer's identity.

If we compare the coefficient of $x^{\mu+k}$ where $\mu \leqslant 2s-k$, the same argument gives

$$\binom{2s}{\mu+k} = \sum_{p=0}^{s-\frac{1}{2}(\mu+k)} \binom{s}{p}\binom{s-p}{\mu-s+p+k} 2^{-\mu+2s-2p-k}. \tag{5}$$

A Note on Geodesic Curvature

The object of this note is to obtain the expression for the geodesic curvature of the orthogonal trajectories of the generators of a ruled surface by a new method.

It is known that the geodesic curvature of a curve $f(u, v) = 0$ is given by

$$\frac{1}{\rho_g} = \frac{1}{H}\begin{vmatrix} E\frac{du}{ds} + F\frac{dv}{ds}, & L \\ F\frac{du}{ds} + G\frac{dv}{ds}, & M \end{vmatrix}, \tag{1}$$

where E, F, G are fundamental magnitudes of the first order, $H = \sqrt{(EG-F^2)}$, and L and M have the significance

$$L = \frac{1}{2}\frac{\partial E}{\partial u}\left(\frac{du}{ds}\right)^2 + \frac{\partial E}{\partial v}\cdot\frac{du}{ds}\cdot\frac{dv}{ds} + \left(\frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u}\right)\left(\frac{dv}{ds}\right)^2 + E\frac{d^2u}{ds^2} + F\frac{d^2v}{ds^2},$$

$$M = \left(\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v}\right)\left(\frac{du}{ds}\right)^2 + \frac{\partial G}{\partial u}\cdot\frac{du}{ds}\cdot\frac{dv}{ds} + \frac{1}{2}\frac{\partial G}{\partial v}\left(\frac{dv}{ds}\right)^2 + F\frac{d^2u}{ds^2} + G\frac{d^2v}{ds^2}.$$

Let the generators of the ruled surface be taken as the parametric curves $v=$ const. and their orthogonal trajectories as the curves $u=$ const. The linear element is given by $ds^2 = Edu^2 + Gdv^2$. Denoting by u the actual distance along a generator, we have $E = 1$. Let γ denote the geodesic curvature of the orthogonal trajectories $u=$ const. and let the suffixes 1, 2 denote differentiation with regard to u, v respectively.

Then along $u=$ const., $du = 0$, $F = 0$, $E = 1$, $ds = \sqrt{G}dv$

$$\therefore \text{ from (1), } \gamma = \frac{1}{\sqrt{G}}\begin{vmatrix} 0 & -\frac{1}{2}\frac{G_1}{G} \\ \sqrt{G} & 0 \end{vmatrix} = \frac{1}{2}\frac{G_1}{G}.$$

We shall obtain this expression for γ by a new method.

If PT is the tangent to the curve $u=$ const. and PM is the normal to the curve in the tangent plane at P, we know that the centre of geodesic curvature of $u=$ const. at P is the unique point on PM whose velocity is perpendicular to PT. Let the co-ordinates of any point on PM be $x+\theta x_1$, $y+\theta y_1$, $z+\theta z_1$. Its displacement is $dx+d\theta.x_1+\theta x_{12}dv$, $dy+d\theta.y_1+\theta y_{12}dv$, $dz+d\theta.z_1+\theta z_{12}dv$. If this is perpendicular to PT we have

$$\Sigma x_2(dx+d\theta.x_1+\theta x_{12}dv)=0$$

or $$\Sigma x_2(x_1du+x_2dv+d\theta.x_1+\theta x_{12}dv)=0$$

or $$G+\theta\,\Sigma x_2x_{12}=0$$

or $$G+\theta\cdot\frac{G_1}{2}=0$$

or $$\theta=-\frac{2G}{G_1}$$

$$\therefore\ \gamma\equiv\frac{1}{\theta}=-\frac{G_1}{2G}.$$

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On a System of Circles Associated with a Symmedian of a Triangle

Three systems of circles are defined here with respect to points on the three symmedians of a triangle and their properties are described. The centres of the members of each system are collinear and the three loci of centres are concurrent at the nine points centre. The nine points circle itself is a common member of the three systems. The circles associated with the points of intersection of the symmedians and the sides of the triangle have additional features and are described separately. The members of each system have double contact with a central conic.

Let ABC be a triangle; A_1, B_1, C_1, the middle points of its sides, D, E, F the feet of the altitudes, S the circumcentre and O the orthocentre of ABC. Let the lines SC_1 and BE intersect in α_1, and SB_1 and CF intersect in α_2. We have then

THEOREM 1.—*If P is any point on AK—the symmedian through A—and if XPY be drawn parallel to AB cutting BC in X and AC in Y and X_1PZ anti-parallel to AC cutting BC in X_1 and AB in Z, the points XX_1YZ are concyclic. If this circle cuts AC, AB again in Y_1, Z_1, then*

(1) *X_1Y_1 is antiparallel to AB and XZ_1 is parallel to AC ;*

(2) *AZY, AY_1Z_1 are isosceles triangles with the base angle A ;*

(3) *the centre of this circle is a point T on $\alpha_1\alpha_2$ such that*

$$\alpha_1T : T\alpha_2 = BX : XC.$$

If the point P varies on AK, we get a system of circles whose centres lie on the line $\alpha_1\alpha_2$. It may be noted that we get the same system of circles by drawing lines parallel to AC and antiparallel to AB, though the individual members do not coincide.

If X is the point of intersection of BC and the symmedian through A, and if lines through X parallel and antiparallel to AC meet AB in Z, Z_1; and the lines parallel and antiparallel to AB meet AC in Y, Y_1 then XYY_1ZZ_1 are concyclic. This circle touches BC at X and we have

THEOREM 2.—

(1) *XY_1Z_1 is isosceles*;

(2) *$XYZZ_1$, XYY_1Z are isosceles trapeziums* ;

(3) $XX : YY_1 : ZZ_1 = \sin\ (A-A) : \sin(B-A) : \sin(C-A)$;

(4) $AZ_1 : ZB = b^2 : a^2$;

$AY : Y_1C = a^2 : c^2$;

$BX : XC = c^2 : b^2$;

(5) *The intersection of AS and the perpendicular at X to BC is the other extremity of the diameter through X of the circle.*

With α_1 as origin and $\alpha_1\alpha_2$ as the x-axis, the equation to the circle of Theorem 1 is of the form

$$(x-\lambda p)^2 + y^2 = \lambda^2 q^2 + (1-\lambda)^2 r^2 - 2\lambda(1-\lambda) g$$

where $p = \alpha_1\alpha_2,\ q = \dfrac{R \sin B}{\sin A},\ r = \dfrac{R \sin C}{\sin A}$

and $g = \dfrac{R^2 \sin B \sin C \cos A}{\sin^2 A}$.

$\alpha_1\alpha_2$ bisects SO and the circle corresponding to N the nine points centre is the ninepoints circle itself. Hence we have:

THEOREM 3.—*The three systems of circles corresponding to the three symmedians have the ninepoints circle as a common member.*

Since the circles in Theorem 3 are of the form $\lambda^2 S_1 + \lambda S_2 + S_3 = 0$ where S_1, S_2, S_3 are respectively of degrees 0, 1 and 2 in x and y, the envelope E of the system of circles, namely $S_2{}^2 = 4S_1S_3$ is of the second degree. The equation to the system of circles can be put in the form

$$E + \{px - g - r^2 - \lambda(p^2 - q^2 - r^2 - 2g)\}^2 = 0.$$

The E-conic is a central conic with $\alpha_1\alpha_2$ as one of its principal axes. We have then

THEOREM 4.—*All the members of a system of circles have double contact with a central conic.*

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On $\sqrt{(a+\sqrt{b}+\sqrt{c})}$

In the synopsis of an article on the minimum number of surd terms possible in $(\sum_1^n p_i\sqrt{a_i})^2$, Mr. A. A. Krishnaswami Ayyangar mentions that a square root of $6-2\sqrt{2}+\sqrt{3}$ does exist in the form $\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}\pm\sqrt{d}$.* The following note starts from this idea.

We have

$$(\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{w})^2 = x+y+z+w+2\left(\sqrt{(xy)}+\sqrt{(zw)}\right)$$
$$+2\left(\sqrt{(yz)}+\sqrt{(xw)}\right)+2\left(\sqrt{(xz)}+\sqrt{(yw)}\right)$$

which reduces to the form $a+\sqrt{b}+\sqrt{c}$ if any one of the expressions in the brackets vanishes.

Let $\sqrt{(yz)}+\sqrt{(xw)}=0$ *i.e.*, $zy=xw$

and let $\sqrt{w}$ be taken with a negative sign.

Putting $x=p$, $y=pq$, $z=pr$, we get $w=pqr$.

Now, $\{\sqrt{p}+\sqrt{(pq)}+\sqrt{(pr)}-\sqrt{(pqr)}\}^2 = p\{1+\sqrt{q}+\sqrt{r}-\sqrt{(qr)}\}^2$

$$= p\{(q+1)(r+1)+2(1-r)\sqrt{q}+2(1-q)\sqrt{r}\} \qquad (1)$$

which identified with $a+\sqrt{b}+\sqrt{c}$ gives

$$\left.\begin{aligned} a &= p(q+1)(r+1)\\ b &= 4p^2(1-r)^2 q\\ c &= 4p^2(1-q)^2 r \end{aligned}\right\} \qquad (a)$$

provided $q<1$, $r<1$ and p, q, r are rational.

We may consider $a+\sqrt{b}+\sqrt{c}$ as the standard form when the surds are taken with the positive sign.

Changing the signs of $\sqrt{q}$ and $\sqrt{r}$ appropriately, we get

$$\sqrt{(a-\sqrt{b}+\sqrt{c})} = \sqrt{p}-\sqrt{(pq)}+\sqrt{(pr)}+\sqrt{(pqr)} > 0 \qquad (2)$$

$$\sqrt{(a+\sqrt{b}-\sqrt{c})} = \sqrt{p}+\sqrt{(pq)}-\sqrt{(pr)}+\sqrt{(pqr)} > 0 \qquad (3)$$

$$\left.\begin{aligned} &\sqrt{(a-\sqrt{b}-\sqrt{c})} = \sqrt{p}-\sqrt{(pq)}-\sqrt{(pr)}-\sqrt{(pqr)} > 0\\ &\qquad \text{only when } \sqrt{r} < \frac{1-\sqrt{q}}{1+\sqrt{q}}. \end{aligned}\right\} \qquad (4)$$

This shows that $a\pm\sqrt{b}\pm\sqrt{c}$ must be positive. In the example cited

$$a=6,\ b=8,\ c=3 \text{ when } p=3,\ q=\tfrac{1}{2},\ r=\tfrac{1}{3}.$$

Hence $\sqrt{(6-2\sqrt{2}+\sqrt{3})} = \sqrt{3}-\sqrt{\tfrac{3}{2}}+1+\sqrt{\tfrac{1}{2}}\ (>0)$.

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* *Math. Student*, 1945, p. 157

Method of Conjugate Surds

We are familiar with the values of $\sqrt{(a+\sqrt{b})}$ and $\sqrt{(a+\sqrt{b}+\sqrt{c}+\ldots.)}$ when the number of surd terms under the radical sign is $n(n+1)/2$, *i.e.*, when the number of surd terms is 3, 6, 10. etc It is not generally known whether it is possible to have square roots (expressible as quadratic surds) of expressions of the form $a+\sqrt{b}+\sqrt{c}$.

As pointed out by Mr. A. A. Krishnaswami Ayyangar in *The Mathematics Student* (1945), $a+\sqrt{b}+\sqrt{c}$ may have a square root in the form

$$\sqrt{x}\pm\sqrt{y}\pm\sqrt{z}\pm\sqrt{w}.$$

We proceed to develop such square roots by a method illustrated below, which we designate as the 'method of conjugate surds'. We utilise one of the appropriate conjugate surds of the given expression and do not make any assumption regarding the number of surds in the square root. Thus the method is entirely different from those adopted in text-books which fail in certain cases, as for example, in the second and third illustrations worked below.

This method is fairly general and is applicable in all cases. It is best described by actual examples worked out.

Illustrations.—

(1) To find $\sqrt{(a+\sqrt{b})}$ $(a>\sqrt{b})$

We have $\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}=\sqrt{\{2a+2\sqrt{(a^2-b)}\}}$,

and $\sqrt{(a+\sqrt{b})}-\sqrt{(a-\sqrt{b})}=\sqrt{\{2a-2\sqrt{(a^2-b)}\}}$

$$\therefore\ \sqrt{(a+\sqrt{b})}=\tfrac{1}{2}[\sqrt{\{2a+2\sqrt{(a^2-b)}\}}+\sqrt{\{2a-2\sqrt{(a^2-b)}\}}].$$

(2) To find $\sqrt{(a+\sqrt{b}+\sqrt{c})}$.

$$\begin{aligned}&\sqrt{(a+\sqrt{b}+\sqrt{c})}+\sqrt{(a-\sqrt{b}+\sqrt{c})}\\&\quad=\sqrt{\{2a+2\sqrt{c}+2\sqrt{(a^2+c-b+2a\sqrt{c})}\}}\\&\quad=\sqrt{(2a+2p+2(q+1)\sqrt{c})},\ \text{say}\end{aligned}$$

[assuming that $\sqrt{(a^2+c-b+2a\sqrt{c})}=p+q\sqrt{c}$].

Similarly, $\sqrt{(a+\sqrt{b}+\sqrt{c})}-\sqrt{(a-\sqrt{b}+\sqrt{c})}=\sqrt{[2a-2p+2(1-q)\sqrt{c}]}$

$$\therefore\ \sqrt{(a+\sqrt{b}+\sqrt{c})}=\tfrac{1}{2}[\sqrt{\{2a+2p+2(1+q)\sqrt{c}\}}+\sqrt{\{2a-2p++2(1-q)\sqrt{c}\}}].$$

Example.—To find $\sqrt{(6-\sqrt{8}+\sqrt{3})}$:

We have $\sqrt{(6-\sqrt{8}+\sqrt{3})}+\sqrt{(6+\sqrt{8}+\sqrt{3})}$

$$\begin{aligned}&=\sqrt{\{12+2\sqrt{3}+2\sqrt{(31+12\sqrt{3})}\}}\\&=\sqrt{(12+2\sqrt{3}+4+6\sqrt{3})}\\&=\sqrt{(16+8\sqrt{3})}\\&=2+2\sqrt{3},\end{aligned}\qquad\text{(i)}$$

and $\sqrt{(6-\sqrt{8}+\sqrt{3})}-\sqrt{(6+\sqrt{8}+\sqrt{3})} = -\sqrt{(8-4\sqrt{3})}$

$$= -(\sqrt{6}-\sqrt{2}) \qquad \text{(ii)}$$

$\therefore \quad \sqrt{(6-\sqrt{8}+\sqrt{3})} = 1+\sqrt{3}-\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}$ from (i) and (ii).

(3) To find $\sqrt{(6+4\sqrt{2}+3\sqrt{3}+2\sqrt{6})}$:

Here the ordinary text-book method fails, for the number of terms in the square root is not 3 but 4 as will be seen and therefore there is no meaning in assuming the square root in the form $\sqrt{x}+\sqrt{y}+\sqrt{z}$. The method of conjugate surds alone will furnish the required square root. We shall use the conjugate surd $6-4\sqrt{2}+3\sqrt{3}-2\sqrt{6}>0$.

$$\{\sqrt{(6+4\sqrt{2}+3\sqrt{3}+2\sqrt{6})}+\sqrt{(6-4\sqrt{2}+3\sqrt{3}-2\sqrt{6})}\}^2$$
$$= 12+6\sqrt{3}+2\sqrt{\{(6+3\sqrt{3})^2-(4\sqrt{2}+2\sqrt{6})^2\}}$$
$$= 12+6\sqrt{3}+2\sqrt{(63+36\sqrt{3}-56-32\sqrt{3})}$$
$$= 12+6\sqrt{3}+2\sqrt{(7+4\sqrt{3})}$$
$$= 12+6\sqrt{3}+2(2+\sqrt{3}) = 16+8\sqrt{3}$$

$\therefore \quad \sqrt{(6+4\sqrt{2}+3\sqrt{3}+2\sqrt{6})}+\sqrt{(6-4\sqrt{2}+3\sqrt{3}-2\sqrt{6})}$

$$= \sqrt{(16+8\sqrt{3})} = 2+2\sqrt{3} \qquad (1)$$

Similarly $\sqrt{(6+4\sqrt{2}+3\sqrt{3}+2\sqrt{6})}-\sqrt{(6-4\sqrt{2}+3\sqrt{3}-2\sqrt{6})}$

$$= \sqrt{(12+6\sqrt{3}-4-2\sqrt{3})} = \sqrt{(8+4\sqrt{3})}$$
$$= \sqrt{2}+\sqrt{6}. \qquad (2)$$

From (1) and (2), by addition and halving,

$$\sqrt{(6+4\sqrt{2}+3\sqrt{3}+2\sqrt{6})} = 1+\sqrt{3}+\tfrac{1}{2}\sqrt{2}+\tfrac{1}{2}\sqrt{6}.$$

St. Philomena's College,
Mysore.

A. K. SRINIVASAN.

On Certain Integrals

In the present note, short methods are given to find the integrals of certain expressions.

1. Let $I_1 = \int \frac{t^2+1}{t^4+1}\,dt$ and $I_2 = \int \frac{t^2-1}{t^4+1}\,dt$.

To find I_1. First method.

$$I_1 = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt = \int \frac{du}{u^2+2} \quad \text{where } u = t-\frac{1}{t}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{2}}\right).$$

Second method.—Let $t = \tan\theta$.

$$\text{Then } I_1 = \int \frac{\sec^4\theta}{1+\tan^4\theta}\,d\theta = \int \frac{d\theta}{\cos^4\theta + \sin^4\theta}$$

$$= \int \frac{d\theta}{\cos^2 2\theta + \frac{1}{2}\sin^2 2\theta} = \int \frac{2\sec^2 2\theta}{\tan^2 2\theta + 2}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan 2\theta}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t\sqrt{2}}{1-t^2}\right)$$

Both forms of the integral I_1 are correct, since they differ only by the constant, $\pi/2\sqrt{2}$.

To evaluate I_2.—Now $I_2 = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2 - 2}\,dt = \int \frac{dv}{v^2-2}$, where $\left(t+\frac{1}{t}\right) = v$.

$$= \frac{1}{2\sqrt{2}}\log\frac{v-\sqrt{2}}{v+\sqrt{2}} = \frac{1}{2\sqrt{2}}\log\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}.$$

The importance of I_1 and I_2 lies in obtaining readily the following integrals for which the usual method, by means of partial fractions, is tedious:

(i) $\int \frac{t^2}{1+t^4}\,dt = \frac{1}{2}(I_1 + I_2)$. (ii) $\int \frac{dt}{1+t^4} = \frac{1}{2}(I_1 - I_2)$.

(iii) $\int \sqrt{\tan x}\,dx = \int \frac{2t^2}{1+t^4}\,dt$, by putting $\tan x = t^2$

$$= 2\int \frac{t^2}{1+t^4}\,dt = (I_1 + I_2).$$

[Integrals (i) and (ii) are given in Hardy's *Pure Mathematics*, Ninth ed., p. 253, where the second form of the integral I_1 is used].

2. Let $I_3 = \int \frac{t^2+1}{t^4+t^2+1}\,dt$ and $I_4 = \int \frac{t^2-1}{t^4+t^2+1}\,dt$

First method.

$$I_3 = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt = \int \frac{du}{u^2+3}, \text{ where } u = \left(t - \frac{1}{t}\right)$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{t^2-1}{t\sqrt{3}}.$$

Second method.

Let $t = \tan\theta$. Then $I_3 = \int \frac{\sec^4\theta}{\tan^4\theta + \sec^2\theta}\,d\theta = \int \frac{d\theta}{\sin^4\theta + \cos^2\theta}$.

$$\therefore\ I_1 = \int \frac{d\theta}{\cos^2 2\theta + \frac{3}{4}\sin^2 2\theta} = \frac{2}{3}\int \frac{2\sec^2 2\theta}{\tan^2 2\theta + \frac{4}{3}}\,d\theta$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{t\sqrt{3}}{1-t^2}.$$

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It can be easily seen that the two forms of I_1 differ only by a constant and therefore both are correct.

$$\text{Now } I_4 = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt = \int \frac{dv}{v^2-1} \text{ where } v = \left(t+\frac{1}{t}\right)$$

$$= \tfrac{1}{2}\log\frac{v-1}{v+1} = \tfrac{1}{2}\log\frac{t^2-t+1}{t^2+t+1}.$$

(i) $\displaystyle\int \frac{t^2}{t^4+t^2+1}\,dt = \tfrac{1}{2}(I_3+I_4)$. (ii) $\displaystyle\int \frac{dt}{t^4+t^2+1} = \tfrac{1}{2}(I_3-I_4)$.

(iii) Similar methods can be employed to find $\displaystyle\int \frac{t^2+1}{t^4-t^2+1}\,dt$ and $\displaystyle\int \frac{t^2-1}{t^4-t^2+1}\,dt$.

[(ii) is found in Hardy's *Pure Mathematics*, Ninth Ed., p. 253, where the second form of I_3 is taken.]

Trichur. C. S. VENKATARAMAN.

The Octagon on a Quadric Surface

Let $A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8$ be an octagon on a quadric S. Then the octagon $A_1 A_4 A_7 A_2 A_5 A_8 A_3 A_6$ lies on another quadric S'.

We prove the above result by showing

(*a*) that the lines $A_1 A_4$, $A_7 A_2$, $A_5 A_8$, $A_3 A_6$ are all intersected by every one of the lines $A_4 A_7$, $A_2 A_5$, $A_8 A_3$, $A_6 A_1$, and

(*b*) that every two lines of either the first or second set are skew.

It will suffice if we prove the relation (*a*) for one of the lines in the first set, say $A_1 A_4$ and (*b*) for one pair of lines, say $A_1 A_4$ and $A_7 A_2$.

The line $A_1 A_4$ is obviously intersected by $A_4 A_7$ and $A_6 A_1$. Also since the sides $A_1 A_2$ and $A_4 A_5$ of the given octagon are generators of opposite systems of the quadric S, they are coplanar and so $A_1 A_4$ and $A_2 A_5$ intersect. Similarly $A_1 A_4$ intersects $A_8 A_3$.

If $A_1 A_4$ and $A_7 A_2$ were coplanar, then the lines of the second set all of which meet them both will be coplanar, which means that the octagon $A_1 A_2 \ldots A_8$ is a plane octagon which is not the case.

Christian College,
Tambaram. P. KESAVA MENON.

Projective Geometry and Harbour Defence

Warships and merchant vessels sheltering in a harbour are liable to attack by enemy submarines as actually happened when H.M.S. *Royal Oak* was sunk while at anchor at Scapa Flow in October 1939—a feat of daring almost rivalled by British submarines which penetrated the mouth of the Elbe and sank a German cruiser. Accordingly, harbours are protected by submarine mines which float at a suitable depth under water moored to the sea bed by a mooring rope. They are of two kinds: one type exploding mechanically when a ship or a submarine bumps against it, while the other is fired electrically from a control station on the shore. Obviously the advantages are with the latter type where the electric cable itself serves as the mooring rope so that when the mine breaks adrift, it is immediately rendered harmless, while all the mines can be rendered harmless by switching off the current. Lastly, it will be possible to devise methods for exploding mines by means of radio signals from the shore, should such a method be preferred.

Whatever the type of mine used, it is necessary to provide a "safety lane" along which surface craft and submarines may ply without running into danger and without dislodging the mines, and the harbour laws in most cases insist that any ship entering or leaving the harbour should be piloted by expert trained pilots who can recognize and steer in these invisible lanes. The mathematical problem which this raises is that of marking a definite but invisible band of small width on the protected water surface in such a manner that the knowledge of the same can be made available to friends and withheld from the enemy. I propose to discuss one solution of the problem suggested by projective geometry.

Taking the surface of the sea within the harbour as approximately a plane, any curve on it can be conceived as the intersection of the plane with a fixed cone whose vertex is above sea level. Accordingly, if we fix a point source of light at a suitable height on the shore, the rays of light emanating from it will serve as generators of our cone. This light will have to be screened off by an opaque plate in which a curve C of small breadth has been cut out, through which the light L will be visible only at those places which make up the safety lane. Thus the safety lane is obtained as the intersection of the sea face with a cone with vertex L and standing on the base C. We know from Mathematics that any given safety lane whatsoever its shape can be obtained from a suitable base curve C on the opaque plate. The curve C may be obtained by using the principle of the camera. As a vessel sails along the safety lane of the harbour, the guiding light L remains visible, but if it strays from the safety lane, the light will become invisible. In actual practice, L should not become invisible but should change colour, say to

green if the vessel has strayed too far to the right and to, say, red if it strays too far to the left. The colours will thus help in bringing the vessel back to safety. Corresponding to the use of lights at night, a modification for use during day time will be a detail to be worked out.

The method of marking out the "safety lane" on a surface described above is not to be considered purely of military significance. Owing to the difference in the level of the sea bed at various places, big ships can only travel along a " safety lane " where the depth is sufficient for their requirements. Hence the need for a pilot tug to act as a guide in the harbour, whose pilot is well conversant with the depths at various points. The method that I have outlined above has the advantage of making the whole process mechanical so that no difficulty need arise owing to the unavailability of trained pilots at any emergency or a failure or mistake of memory on his part. A second advantage which is claimed for the device described here is that if it is suspected that enemy vessels or submarines are trying to take advantage of the method described above of steering along the safety lane, all that is necessary is to switch off the light L or better still to change the plate containing the curve C. The " safety lane " marked out by the substituted plate (wrong plate) will not be really safe and will lead the enemy vessels into danger. Lastly, while more elaborate methods may be used to secure the same objective, the method outlined above should be cheap and simple and easy to handle.

Andhra University,
Waltair.

A. NARASINGA RAO.

REVIEWS

Introduction al Algebra Vectorial

POR PEDRO PI CALLEJA

(Universidad Nacional de Cuyo, Buenos Aires, 1945)

In this book of 137 pages (in Spanish) the author gives a well-planned treatment of Vectors and Tensors of the second order and some of their illustrative applications. The addition of Vectors and their multiplication by scalars are defined by a set of 15 postulates, including the dimensional postulate, while 3 more are required for defining the scalar product. These are sufficiently broad to permit of their application later to function spaces. The volume determined by n Vectors $A_1, A_2, \ldots, A_n$ in n-space is defined as a scalar $D(A_1, A_2, \ldots, A_n)$ which is (i) unaffected by replacing say A_j by $A_j + A_k$, (ii) which is multiplied by λ when A_j is replaced by λA_j and (iii) such that $D(U_1, U_2, \ldots, U_n) = 1$ where the U's are unitary vectors, and then the usual properties of determinants are proved for $D(A_1, \ldots, A_n)$. After a treatment of the external product, the author considers briefly abstract metric spaces as met with in modern Mathematical Physics. Tensors are considered as dyadics, as linear operators, and as matrices. The book is divided into 6 chapters with examples at the end of each and an excellent bibliography of connected literature.

A. N. RAO.

Illustrative Analytical Geometry of Two Dimensions

BY BRIJ LALL MANOCHA, Hoshiarpur, Lahore

(Atma Ram & Sons, 1946, pp. 127, Price Rs. 5)

This is a collection of 238 problems in Analytical Geometry, with their solutions or hints for their solution. The topics covered are circular points, homogeneous co-ordinates, tangential co-ordinates, homography and involution, invariants, etc. The problems and solutions take up the whole book except the first six pages which are devoted to explanations of the principle of duality, of reciprocation, of " Grassman's Symbols ", and elements at infinity.

Judged even by the modest standards expected of a guide for examinations, the book is open to serious criticism. The topics covered by the questions are not arranged in any natural order. Thus metrical problems come somewhere in the middle, homography and involution at the end, tangential co-ordinates and reciprocation are separated by other sections, while the first question in the whole collection is "Obtain the equation of the line at infinity. Find the co-ordinates of the circular points "—and this under a note which says, "In this section we deal with Cartesian co-ordinates"! The answer, however, is not as absurd as the question, and gives $z = 0$ as the equation of the line at infinity, the author apparently having no suspicion that he is using homogeneous co-ordinates. The explanatory part at the beginning is all too brief and there is no attempt at a careful formulation of basic concepts. The use of Grassman's symbols is an intrusion without

a clear explanation of their meaning and serves no useful purpose. While the students for whom it is intended may get some help in solving the usual examination questions, one feels that a fine opportunity has been lost by failure in planning properly, and in presenting the fundamentals of the subject in their proper perspective.

A. N. RAO.

B.A. and B.Sc. Algebra

BY S. SURYANARAYANA IYER AND A. SANTIAGO

(Vol. I, pp. 85, Price Rs. 2; Vol. II, pp. 188, Price Rs. 4; Vol. III, pp. 192, Price Rs. 3–8–0. First Edition, 1947)

The object of the authors seems to be fairly achieved in presenting to the students the necessary examination requirements in Algebra, starting from inequalities and ending with determinants. The authors have drawn their inspiration from Barnard and Child's *Higher Algebra* though they have omitted a good deal of the difficult parts with an apologetic statement that the proofs are beyond the scope of their work and a pathetic appeal to intuition.

Vol. I contains three chapters dealing in order with inequalities, sequences and limits, and convergence and divergence of series. Vol. II takes us on in six chapters through partial fractions, the binomial theorem, homogeneous products, the exponential theorem, the logarithmic series and summations. Vol. III has nine chapters, of which the first six deal with the theory of equations, the next two with continued fractions and its application to the linear indeterminate equation and the last with the theory of Determinants. It is notorious that the simplest basic theorems in determinants are the most difficult to prove to the beginner with rigour and clarity and it is no wonder that the authors have not succeeded in their attempt to do the nearly impossible. Half figurative phrasings like 'the changed determinant is equal to the given determinant with the sign changed', 'in the new determinant this term is evidently a term' are not conducive to clarity.

The method of exposition followed is the mechanical three-step routine of book-work, worked examples and exercise, which goes on through eighteen chapters. Unwarranted abbreviations $|||^{ly}$ and eqn. are freely employed smacking of classroom black-board work and there are minimal verbal elucidations from step to step. The worked examples which are the staple food of the B.A. and B.Sc. students are printed in very small type difficult to read. The book provides answers and corrigenda though the latter require supplementing. Though there is good spacing in between lines, the printing is not quite satisfactory; the symbol for 'tends to' takes several avatars. We look forward to a better and cheaper second edition.

A. A. K.

A Text-book on Co-ordinate Geometry

BY GORAKH PRASAD AND H. C. GUPTA

(Pp. 327, 1947, Price Rs. 5. Pothisala, 2 Lajpat Road, Allahabad)

This text-book is intended for the B.A. and B.Sc. students of Indian Universities and covers fully in the traditional manner all the topics of Cartesian Geometry of two dimensions—the straight line, the circle, the parabola, the ellipse, and the hyperbola including the general equation of the second degree and the polar equations. Change of axes and oblique axes are also treated but applied very sparingly. There is a miscellaneous chapter at the end dealing with confocal conics, and conic-pairs in several orders of contact. The authors think that it is not possible to distinguish in a diagram two pairs of conics—one pair having contact at a point with the other points of intersection imaginary, while the other pair have four-pointic contact. The distinction can be visualised if we draw any straight line through the point of contact, say P, to cut the conics again at Q and R. If we draw the tangents at Q and R, they will meet on the tangent at P only in the case of four-pointic contact.

There is a copious supply of graded exercises for which hints are provided and answers given when necessary. The book-work is clearly presented without any pedantic digressions. On the whole, the text-book is quite sober and does not take risks by entering upon niceties or controversial points. On one occasion, however, the authors try to diverge from the beaten track when they solve the perplexity in deciding on which side of the tangent at the vertex a general parabola lies by substituting the co-ordinates of a known point in the equation of the tangent and noting the relative positions of the point and the parabola. But a better method is to write down the equations of transformation for the general parabola to the standard form and note the new co-ordinates of the erstwhile origin, which will give immediately the orientation of the new axes for the standard parabola.

The book is well printed and bound and the diagrams are good. It is a neat and handy volume which should make a ready appeal to examination-going students.

A. A. K.

An Introduction to the Theory of Equations

BY RAM BEHARI AND HANS RAJ GUPTA

(Pp. 169, 1947. Price Rs. 7–8–0. S. Chand & Co., Delhi and Lahore)

This short book is a fairly complete introduction to the methods of solving general equations of the third and fourth degrees and computing the roots of numerical equations of a higher order under certain favourable conditions like commensurability and multiplicity. The necessary theory to lead up to this includes the study of symmetric functions of the roots of an equation, transformation of equations and the location and the nature of the roots of an equation by

Descarte's Rule of Signs, Newton's theorem on superior limits of roots, and Sturm's functions. The authors have taken special pains to work out a large number of numerical examples to illustrate the algorithms employed. Special stress is laid on the practical and computational aspect of the theory of equations and the authors have shown elegant methods of layout of the numerical steps. In every chapter there are many numerical exercises with answers side by side.

Probably one defect of the book is that the proofs of difficult theorems like Sturm's theorem are unduly curtailed without adequate explanations. There are certain dogmatic assertions which might have been elucidated even at the expense of a little prolixity. Rolle's theorem is nowhere mentioned, while binomial equations are finished off as an application of Demoivre's theorem. By way of change from traditional matter we have Biswas's method of forming the equation whose roots are the squared differences or semi-sum of the roots of an equation and reproduction of two useful and relevant articles from *The Mathematics Student* and *The American Mathematical Monthly.*

The book ends with questions on theory of equations from six Delhi University examination papers and sixty miscellaneous problems, for which answers are not provided.

As a compendious account of the computational aspect of the theory of equations, the book deserves all encouragement at the hands of University teachers and students. Our Honours students lack computational training and this book should be thoroughly worked through by them.

It is no use blaming the Indian Press for its lack of technical efficiency in printing $\sqrt[n]{y}$ as $n\sqrt{y}$. If we industrialise rapidly, we can avoid such blemishes, but typographical and punctuation errors could have been easily avoided with a more careful proof-reading.

The price of the book is unfortunately a little too high for the poor Indian scholar to invest on a moiety of Pure Mathematics.

A. A. K.

Calculus

BY G. L. CHANDRATREYA AND L. V. GURJAR

(Pp. vi, 313, xxxi, 1948. Price Rs. 5. Ideal Book Services, 885, Sadashivpeth, Poona 2)

This book is meant to be a first course in Calculus but it transcends the needs of students of the Intermediate standard in Indian colleges. It has a few new and good features. Graphs of functions like $y^2 = x^2 (a + x)/(a - x)$ are introduced even in the first chapter to widen the outlook of students regarding the nature of a function. All the elementary mathematical functions—algebraical, trigonometrical, exponential and logarithmic are duly differentiated and integrated and their geometrical and mechanical applications are fairly fully discussed. Maxima, minima, areas and volumes are treated with adequate clarity and completeness for

an elementary stage. It has not been thought improper to give the enterprising pupil a peep into 'improper' integrals also. Nor are the needs of the backward student forgotten and a list of formulæ in Trigonometry precedes the study of trigonometrical derivatives. But general elementary theorems on limits are proved by the introduction of several epsilons, all becoming ultimately suitably small and leading to the desired result. This process may not be deemed quite as simple or æsthetical as the usual method which utilises appropriate fractions of a single epsilon.

The book abounds in numerous hard examples, worked and unworked and closes with a few University papers (the University though unspecified is Bombay) dating perhaps curiously backwards from 1948 to 1945.

The printing and get-up of the book are commendable, though the price may prove a damper to the enthusiastic scholar who wishes to acquaint himself with more than elementary knowledge of Calculus.

A. A. K.

Theoretical Aerodynamics

BY L. M. MILNE-THOMSON, Professor Royal Naval College

(Macmillan & Co., London, 1948, pp. 363 + xix, price 40/- net)

Aerodynamic studies are still in their infancy in this country, and it is hoped that the publication of this excellent book will stimulate a greater interest in this subject whose importance, at the present time can hardly be exaggerated.

The central problem of Theoretical Aerodynamics is that of studying the flow of air round an aircraft and the forces acting on it, with a view to the design of airoplanes with desirable characteristics. The phenomenon is one of great complexity and demands various simplifying assumptions, in order to enable a solution to be obtained in the present state of our knowledge. The assumptions usually made are that air may be treated as (i) an incompressible, (ii) "ideal" (non-viscous) fluid. For aircraft whose velocity is small compared with the velocity of sound in air, assumption (i) can be theoretically justified, while (ii) is found to be true to a high degree except in the immediate neighbourhood of (the boundary layer) the aircraft which is the seat of highly complex forces whose integrated effect might be taken into account by postulating suitable "vorticity distributions". If the further assumption (iii) be made that the span of the wing—the aircraft being idealised into a moving wing—is infinite, the problem is reduced to two-dimensional potential motion, where the usual representation by means of complex functions becomes available. It is found tha the motion is indeterminate unless the "circulation" around the wing section is known, this being capable of any value. It was the Russian Mathematician Joukowski who in 1910 obtained the "Joukowski Profile" as the conformal image of a circle and pointed out how the circulation Γ and hence the lift $\rho\Gamma V$ could be uniquely determined by the requirement that the velocity at the sharp trailing edge should be finite. The two-dimensional theory first studied by Kutta and Joukowski, and generalised by R. Von

Mises is developed in Chapters III to VIII of the book under review, of which the last chapter is denoted to "thin airfoils". A theorem due to the author himself "the circle theorem" (p. 84), is found to be of the utmost use in determining the complex potential of any two-dimensional flow as perturbed by the introduction of a circular cylinder.

The two-dimensional theory accounts for the "lift" but not for the drag of an airfoil. The 3-dimensional theory, *i.e.*, the theory of wings of finite span was inaugurated by F. W. Lanchester and was later enriched by L. Prandtl (1918) and E. Trefftz, the central feature here being the study of the trailing vortices in the wake of the aircraft and the velocity field induced by them. This accounts for that part of the drag not due to skin friction and gives the famous integro-differential equation of Prandtl for determining the distribution of circulation along the span. The three-dimensional theory, including propellers and wind tunnel corrections is given in Chapters IX to XIV.

The all-important question of supersonic (and also subsonic) flow, which has become a reality with jet engines is touched in Chapter XV where steady flow in an ideal compressible fluid past a slender general profile with sharp leading and trailing edges is discussed by means of the linearised theory.

Chapters XVI to XVIII are devoted to various flight problems including Lanchester's phugoid motion and to questions of stability. The first two chapters in the book deals with preliminary notions and Bernoulli's theorem, while the last one is devoted to a brief treatment of vector analysis to equip the reader to follow the vector methods used in the book. A collection of 300 examples, most of which were specially framed for the occasion, and an index at the end, add to the utility of the book. Besides a large number of diagrams, there is one photograph which shows the helical vortices in the wake of the propeller.

The book is prepared on lines similar to and is a companion volume to the author's *Theoretical Hydrodynamics* and is thus eminently suitable for use both as a text-book and by those who wish to have a first course in this fascinating study.

A. N. RAO.

Co-ordinate Geometry for Intermediate Classes

BY K. RAJAGOPALAN, *Lecturer, Sri Venkateswara College, Tirupati*

(Published by the author, 1948, 118 pp., Price Rupee one and annas four)

This booklet deals with the analytical geometry of the straight line and the circle. There is a fairly clear presentation of the usual material, with a large number of worked examples (nearly 50) illustrative of the methods discussed, besides exercises for the student and questions from the Madras University papers. A distinguishing feature of the book is that separate diagrams are drawn corresponding to different relative positions of the points and lines or different signs of the constants, to show that the resulting formula is the same in all cases. This is all to the good, since it is only at the intermediate stage that confidence

in the generality of the algebraic method and in the invariance of its results should be built up. The printing and get up are fairly good, and the book should prove very useful for those preparing for the Intermediate examination.

On the other hand it has to be stated that, though copious, the examples worked out are of a kind involving only a mechanical application of the rules, and that no attempt has been made to link up the subject with the geometrical experience or intuition of the learner. Even where several solutions appear, the author does not take the trouble to point out how they arise and should be expected. Cases of proving known geometrical results by elegant analytical methods are rare, and owing to the large number of numerical examples, there is little opportunity for the learner to choose his axes in the most advantageous manner. So far as the reviewer has noticed, there is no attempt at interpreting $S > 0$ or $S < 0$, where S is the equation of a straight line or circle. Such an interpretation is independent of the formula for the distance of a point from a line, and the learner should know not only the locus corresponding to $S = 0$ but also the meaning of the above inequalities.

A. N. RAO.

ANNOUNCEMENTS AND NEWS

The following persons have been admitted as members of the Indian Mathematical Society:—

(i) Paraschand Sanghi, Esq., M.SC., Lecturer in Mathematics, S. K. N. I. R. College, Jobner, Jaipur City.

(ii) Miss Kusum Hari Gore, B.A. (Hons.), B.T., Teacher, M. H. High School, Mahajisi Lane, Thana.

(iii) Abul Kazim Rizvi, Esq., M.A., Lecturer in Mathematics, Muslim University, Aligarh.

(iv) R. D. James, Esq., Professor of Mathematics, University of Vancour, Vancour British Columbia, Canada.

(v) Miss R. De, Lecturer in Mathematics, Women's College, Cuttack.

(vi) B. N. Mukerjee, Esq., M.A., Professor of Mathematics, Scottish Churches College, Calcutta.

(vii) Haridas Bagchi, Esq., M.A., Ph.D., Lecturer in Pure Mathematics, Calcutta University, 124/4, Manicktolla Street, Beadon Street Post Office, Calcutta.

(viii) Miss Kamalamma, M.A., Lecturer in Mathematics, Maharani's College, Bangalore.

(ix) Onkar Prasad Mital, Esq., M.SC., Pukhraj Bhavan, Beechla Pal, Ajmer.

(x) C. S. Venkataraman, Esq., M.A., M.SC., Department of Mathematics, Kerala Varma College, Trichur.

The following have paid the life-composition fee of Rs. 150 and have become members of the Society for life:—

(i) H. G. S. Sarma, Esq., Professor of Mathematics, Lingaraj College, Belgaum.

(ii) D. K. Sen, Esq., Professor of Mathematics, Rajaram College, Kolhapur.

(iii) G. S. Mahajani, Esq., Vice Chancellor, Rajaputana University, Jaipur.

(iv) B. Rama Murti, Esq., Statistician, Labour Bureau, Ministry of Labour, Government of India, Simla.

A new Journal, Communications on Applied Mathematics, has been announced by the Institute for Mathematics and Mechanics at New York University. It will be issued quarterly from January 1948.

Our readers will be glad to learn that Professor O. G. Tietjens, has accepted an appointment as the Head of the Department of Aeronautical Engineering in the Indian Institute of Science, Bangalore.

Dr. B. R. Seth of the University of Delhi has accepted an invitation as Visiting Professor of Applied Mathematics for the year 1949 by the University of Iowa and will be in residence at the Iowa State College, Ames, Iowa, U.S.A.

Prof. D. D. Kosambi of the Institute of Fundamental Research, Bombay, is on a visit to the U.S.A. on a Fellowship from the UNESCO to work on the new types of electronic calculating machines for setting up a modern calculating laboratory in Asia. He has been also invited as a visiting professor by the Chicago University.

The University of Madras has awarded the Degree of Doctor of Science on P. Kesava Menon, Lecturer in Mathematics, Madras Christian College, Tambaram.

Among the distinguished mathematicians who visited India at the end of 1948 were Prof. S. Chapman of Oxford and Prof. W. Sierpinski of Warsaw. Prof. Chapman is giving a course of lectures on Geomagnetism at Bombay, and Prof. Sierpinski will be giving a course of lectures in the Lucknow University.

Prof. G. H. Hardy, distinguished English Mathematician and one of the Honorary Members of the Society, died on December 1st, 1947. An obituary appears in this issue.

Prof. A. N. Whitehead of Harvard University died on December 30th, 1947.

The Society has accepted the invitation of the Madras University to hold its next biennial Conference at Madras in December 1949.

The Seventh International Congress of Applied Mechanics

The Seventh International Congress of Applied Mechanics was held at the Imperial College, London, from the 5th of September to the 11th of September 1948 under the presidentship of Sir Richard Southwell. About 800 delegates from all parts of the world attended it, and these included distinguished workers like Th. Von Karman, Sir Geoffry Taylor, Prof. S. Timoshanko, Prof. H. L. Dryden, Prof. W. Prager. Dr B. R. Seth of Delhi University atter ded the session.

On the 5th of September the Governing Body of the Imperial College invited the delegates to an evening party. From the 6th to the 11th of September there were on each day two technical sessions and at least three general lectures. The Congress was divided into four sections in the following manner:—

SECTION I. Elasticity and Plasticity.

SECTION II. Aerodynamics, Hydrodynamics and Meteorology.

SECTION III. Thermo-dynamics, Heat Transfer, etc.

SECTION IV. Vibrations, Lubrication, Electronics, etc.

About 200 papers were accepted to be read in the four sections. Sections I and II had each a sub-section. On the average about three papers were read and discussed in each session of about three hours. This is a very good practice worthy of adoption by scientific bodies. A fairly comprehensive summary of all accepted papers was available before the session began and in some cases even complete papers were given to the delegates.

In the first technical session the discussion on "Finite Elastic Deformation" was opened by Prof. L.M. Milne-Thompson who referred to the work of Dr. B. R. Seth and Prof. F. D. Murnaghan, and explained how it could be extended to aelotropic bodies. Dr. B. R. Seth, commenting on the paper, gave examples where the theory as applied to aelotropic bodies, had been successfully worked out by him. Mr. R. S. Rivlin, Prof. H. Bradenburger, Dr. H. K. Swainger and Prof. A. Sigorini also took part in the discussion.

Dr. B. R. Seth read a paper on the "Bending of cylindrical shells with vertical walls" in which he showed that the solution for the bending of a supported rectilinear plate under uniform pressure could be made to depend on the corresponding torsion solution.

Among the important papers was one by Sir Geoffery Taylor on "Swirl Atomisers" showing where the engineers had gone wrong in the calculation of the boundary layer. Th. Von Karman and Prof. C. C. Lin spoke on the theory of Turbulence and Prof. H. L. Dryden on the design of wind-tunnels.

Among the general lectures delivered at the Congress were:—

(1) "Recent developments in the mathematical theory of plasticity" by W. Prager.

(2) "Cinema aid in teaching mechanics", by L. Couffignal.

The technique adopted in the second was very good. It was a sound film on the kinematics of the Universal milling machine.

On the social side there was a reception at the Senate House by the Vice-Chancellor of the University of London, an informal dance, Reception by the Lord Mayor at the Guildhall and the Congress Banquet.

Delegates visited the National Physical Laboratory, College of Aeronautics, National Gas Turbine, Royal Air-Craft Establishment and General Electric Co.

CONNEMARA 12 MAY 1949 MADRAS

OBITUARY

G. H. Hardy

Godfrey Harold Hardy, an honorary member of our Society and the most famous British mathematician of his day, died in England, on 1st December 1947, at the age of sixty-eight. In India he will be remembered not only for his mathematics but also for his discovery of S. Ramanujan and all that he did to Ramanujan. It may be true as Hardy* himself said that in teaching Ramanujan some modern mathematics he learnt more from Ramanujan than Ramanujan from him; elsewhere† Hardy says that " playing the game of which he (Ramanujan) knew the rules, he could give any mathematician in the world fifteen "; yet I think it should be conceded that it is improbable that Ramanujan would have been the famous mathematician that he became but for his association with Hardy. Besides Ramanujan many of the mathematicians in India owe much to Hardy, directly or indirectly.

Professor Hardy had in abundance those qualities which enable a person to be a founder of a school of mathematics. He gave generously of his time to look into any decent original mathematical paper that any one might send him; he would, if necessary, discuss with and advise the authors on those papers and help them in preparing the manuscripts for publication. Indeed I remember being told that the press copy of almost all—perhaps all—the papers that Ramanujan published while he was in England were in the handwriting of Hardy. He would forbid any thankful acknowledgment in print of any help received from him unless it was of a substantial kind like, for instance, supplying the proof of a difficult point in the establishment of the main theorem in a paper.

He had a flair for collaborating with many mathematicians, British and foreign. His influence on the British mathematics of his day was very considerable. The style of his lectures, published papers and books was such that it was difficult for any of his bright students not to be infected by Hardy's enthusiasm for the subject; both when listening to his lectures and reading his publications one had the feeling that at every stage Hardy took the audience fully into his confidence. A theorem was often looked at from more than one point of view, and two or even three proofs were given; attention was drawn to points of interest in the theorem and crucial steps in its proofs. The reader was told of various related problems which Hardy had not yet solved. It was not enough for Hardy to be a discoverer of important and beautiful theorems; he wanted to make it as easy as he could for others to share in full the pleasures of contemplating new and old results.

* *Collected Papers of Srinivasa Ramanujan*, pp. xxx–xxxi.

† *The American Mathematical Monthly*, Vol. XLIV, 1937, pp. 139.

In his Presidential Address* to the Mathematical Association, in the year 1926, he said "....the subject (mathematics) has after all been the one great permanent happiness of my life. My devotion to Mathematics is indeed of the most extravagant and fanatical kind; I believe in it, and love it, and should be utterly miserable without it....." What he said then was true of him till the very end, and to the last a steady stream of papers came from his pen.

He published some ten books four of which were in collaboration with others. His first book, *A Course of Pure Mathematics* was published more than forty years ago and is still very popular in Britain and India. It is a great and remarkable book which was written with the zeal of an evangelist; the object was to introduce the college student in Britain to decent mathematical analysis and to rouse his enthusiasm for the subject. Hardy held the view that mathematics was a badly taught subject and that otherwise it would be one of the most popular subjects.

He loved lecturing and was happy in front of a blackboard; his lectures were completely supported by written notes. Few could equal him in his apt choice of words, phrases and imagery with which he could hit off a perfect explanation of a difficult or complicated or subtle argument. He was a master of English prose of a very high order and his moving book *A Mathematician's Apology* is a testimony to this.

This brief notice is not suited to give an account of the mathematical works of Professor Hardy, nor am I competent to pass judgment on them. His chief contributions to the mathematics of his day appear to be:—

(i) Studies in the theory of functions of a complex variable, chiefly of the properties of the Riemann Zeta function, and their applications to problems such as the distribution of prime numbers. Hardy was the first to prove that an infinity of the complex zeros of the zeta function have their real part equal to $\frac{1}{2}$.

(ii) Converse theorems on summability.

(iii) The discovery of a method to tackle some of the most difficult problems in additive number theory. An account of the method first occurs in a famous paper of Hardy and Ramanujan on partitions of numbers. It was a refinement of this method that enabled P. Vinogradow to prove that all odd numbers from a certain point onwards are sums of three odd primes.

(iv) Theorems on Fourier Series.

Inequalities, evaluations of definite integrals and Diophantine approximations always interested him. He published scores of papers under the title *Notes on Integral Calculus.*

Waltair. T. VIJAYARAGHAVAN.

* *Mathematical Gazette.*

INDIAN MATHEMATICAL SOCIETY

STATEMENT OF ACCOUNTS FOR 1944

RECEIPTS

	Rs.	A.	P.
Opening Balance—			
Fixed Deposit with Indian Bank	10,000	0	0
Current Account with Indian Bank	329	10	11
Savings Bank Account with Indian Bank	3,781	2	0
Savings Bank Account with Triplicane Fd.	1,752	11	10
Cash with Librarian	507	13	6
Subscriptions—			
Life-Members	225	0	0
Members	584	0	0
Subscribers	781	8	0
Sale of Back Numbers, etc.	750	11	4
Interest Receipts	92	1	1
Grants in Aid—			
Bombay University	200	0	0
Madras University	150	0	0
Annamalai University	192	3	9
Total	19,346	14	5

EXPENDITURE

	Rs.	A.	P.
Editorial	23	6	0
Secretary	107	3	0
Paper	62	10	3
Printing	904	3	3
Bank Charges	6	14	0
Library	170	1	0
Closing Balance—			
Fixed Deposit	10,000	0	0
Current Account with Indian Bank	1,074	5	8
Savings Account with Indian Bank	4,564	14	7
,, ,, Triplicane Fund	1,840	8	2
Cash	592	12	6
Total	19,346	14	5

Checked and found correct.

A. A. Krishnaswami Ayyangar,
Hon. Auditor.

A. Narasinga Rao,
Hon. Treasurer.

INDIAN MATHEMATICAL SOCIETY

STATEMENT OF ACCOUNTS FOR 1945

RECEIPTS

	Rs.	A.	P.
Opening Balance—			
Fixed Deposit	10,000	0	0
Current Account with Indian Bank	1,074	5	8
Savings Account with Indian Bank	4,564	14	7
” ” Triplicane Fund	1,840	8	2
Cash	592	12	6
Subscriptions—			
Life-Members	450	0	0
Ordinary Members	1,806	0	0
Subscribers, Sale, etc.	1,428	7	6
Interest Receipts	292	1	2
Grants in Aid—			
Madras University	150	0	0
Bombay University	100	0	0
Osmania University	100	0	0
Rockfeller Grant	450	0	0
Donations to the Publication Fund—			
*Travancore University	1,000	0	0
From Members	48	0	0
Narasinga Rao Medal Endowment	1,000	0	0
Total	24,897	1	7

EXPENDITURE

	Rs.	A.	P.
Editorial	89	15	9
Librarian	200	3	6
Secretary	132	0	6
Miscellaneous	3	6	6
Printing	712	4	3
Closing Balance—			
Fixed Deposit	10,000	0	0
Current Account with Indian Bank	1,651	12	8
Savings Account with Indian Bank	6,026	1	9
Triplicane Fund	1,840	8	2
Co-operative Urban Bank, Chidambaram	621	5	6
Publication Fund	1,048	0	0
Narasinga Rao Medal Fund	1,000	0	0
Cash	1,571	7	0
Total	24,897	1	7

Checked and found correct.

A. A. Krishnaswami Ayyangar,
Hon. Auditor.

A. Narasinga Rao,
Hon. Treasurer.

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